~ Syllabus ~

MODULE - 1

Limit, continuity for functions with severable variables, partial derivatives, total derivative, Maxima, minima and saddle points; Method of Lagrange multipliers, Multiple Integration: double and triple integrals (Cartesian and polar), Change of order of integration in double integrals, Change of variables (Cartesian to polar), Applications of double and triple integrals to find surface area and volumes.

MODULE - 2

Sequence and series, Bolzano Weirstrass Theorem, Cauchy convergence criterion for sequence, uniform convergence, convergence of positive term series: comparison test, limit comparison test, D'Alembert's ratio test, Raabe's test, Cauchy root test, p-test, Cauchy integral test, logarithmic test, Alternating series, Leibnitz test, Power series, Taylor's series, Series for exponential, trigonometric and logarithmic functions.

MODULE - 3

Exact, linear and Bernoulli's equations, Euler's equations, Equations not of first degree: equations solvable for p, equations solvable for y, equations solvable for x and Clairaut's type.

MODULE - 4

Second and higher order linear differential equations with constant coefficients, method of variation of parameters, Equations reducible to linear equations with constant coefficients: Cauchy and Legendre's equations.

PUNJAB TECHNICAL UNIVERSITY QUESTION PAPERS

UNIVERSITY QUESTION PAPER, DEC.–2020

SECTION – A

Q 1. Show that the limit for the function $f(x, y) = \frac{x^2+y^2}{x^2-y^2}$ **does not exists as** $(x,y) \to (0,0)$.

Ans. *Refer to Module 1 Q.No.101 on Page No. 61*

Q 2. Evaluate the integral $\int_{-1}^{1}\int_{0}^{z}\int_{x-z}^{x+z} dydxdz$.

Ans. *Refer to Module 1 Q.No. 102 on Page No. 61*

Q 3. Check the convergence of the following sequences whose nth term is given by $a_n = \left(\frac{3n+1}{3n-1}\right)^n$.

Ans. *Refer to Module 2 Q.No.53 on Page No. 91*

Q 4. State Cauchy Integral test for convergence of a positive term infinite series.

Ans. *Refer to Module 2 (Definition of Cauchy Integral) on P.No. 66 & Q.No. 19 on P.No.72*

Q 5. Write down the Taylor's series expansion for sin x about $x = \frac{\pi}{2}$.

Ans. *Refer to Module 1 Q.No. 103 on Page No. 61*

Q 6. Solve by reducing into Clairaut's equation : $p = \log(px - y)$, **where** $p = \frac{dy}{dx}$.

Ans. *Refer to Module 3 Q.No. 20 on Page No. 103*

Q 7. Solve the differential equation $\frac{dy}{dx} + y \cot x = x \operatorname{cosec} x$

Ans. *Refer to Module 3 Q.No. 59 on Page No. 125*

Q 8. Determine whether the differential equation is exact $(x^2+y^2+2x)\,dx + 2ydy$

Ans. *Refer to Module 3 Q.No.60 on Page No. 126*

Q 9. Solve the differential equation $\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 0$

Ans. *Refer to Module 4 Q.No.60 on Page No. 163*

Q 10. Find Particular integral for $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = e^{-x}$

Ans. *Refer to Module 4 Q.No. 61 on Page No. 164*

SECTION – B

Q 11. Using Method of Lagrange Multipliers, find the maximum and minimum distance of the point (3, 4, 12) from the sphere $x^2 + y^2 + z^2 = 1$.

Ans. *Refer to Module 1 Q.No. 104 on Page No. 62*

Q 12. Solve by changing order of integration : $\int_0^a \int_y^a \frac{x}{x^2+y^2} dxdy$, a is any positive constant.

Ans. *Refer to Module 1 Q.No. 105 on Page No. 63*

Q 13. For what value(s) of x does the series converge (i) conditionally (ii) absolutely?

$x - \frac{x^2}{\sqrt{2}} + \frac{x^3}{\sqrt{3}} - ......$ to ∞. Also find the interval of convergence.

Ans. *Refer to Module 2 Q.No. 45 on Page No. 85*

Q 14. Solve the differential equation :

$(xy^3 + y)\, dx + 2\,(x^2y^2 + x + y^4)dy = 0$

Ans. *Refer to Module 3 Q.No. 28 on Page No. 106*

Q 15. Solve the differential equation $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = xe^{3x} + \sin 2x$.

Ans. *Refer to Module 4 Q.No. 62 on Page No. 164*

SECTION – C

Q 16. (a) Check the convergence of the series $\sum_{n=2}^{\infty} \frac{\sqrt{n+1}-\sqrt{n}}{n^{3/2}}$.

Ans. *Refer to Module 2 Q.No. 54 on Page No. 91*

(b) Find by double integration, the area lying inside the circle $r = a \sin\theta$ and outside the cardiode $r = a\,(1 - \cos\theta)$.

Ans. *Refer to Module 1 Q.No. 106 on Page No. 64*

Q 17. (a) Solve the differential equation $\frac{dy}{dx} + \frac{x}{1-x^2} y = x\sqrt{y}$.

Ans. *Refer to Module 3 Q.No. 61 on Page No. 126*

(b) Solve the differential $xyp^2 - (x^2 - y^2)\, p + xy = 0$, where $p = \frac{dy}{dx}$.

Ans. *Refer to Module 3 Q.No. 62 on Page No. 127*

Q 18. (a) Solve by Method of Variation of parameters $\frac{d^2y}{dx^2} + y = \sec x$.

Ans. *Refer to Module 4 Q.No. 20 on Page No. 139*

(b) Solve $(1+x)^2 \frac{d^2y}{dx^2} + (1+x)\frac{dy}{dx} + y = \cos \ln(1+x)$.

Ans. *Refer to Module 4 Q.No. 63 on Page No. 165*

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Module 1

Syllabus

Limit, continuity for functions with severable variables, partial derivatives, total derivative, Maxima, minima and saddle points; Method of Lagrange multipliers, Multiple Integration: double and triple integrals (Cartesian and polar), Change of order of integration in double integrals, Change of variables (Cartesian to polar), Applications of double and triple integrals to find surface area and volumes.

BASIC CONCEPTS

PARTIAL DERIVATIVES

If z be a function of two independent variables x and y and it is written by $z = f(x, y)$
Then the partial derivatives of z w.r.t. x and y of Ist order and IInd order are given by

$$f_x = \frac{\partial f}{\partial x} = \frac{\partial z}{\partial x} = \underset{\delta x \to 0}{Lt} \frac{f(x+\delta x, y) - f(x, y)}{\delta x}$$

$$f_y = \frac{\partial f}{\partial y} = \frac{\partial z}{\partial y} = \underset{\delta y \to 0}{Lt} \frac{f(x, y+\delta y) - f(x, y)}{\delta y}$$

partial derivative of Ist order

$$\frac{\partial^2 f}{\partial x^2} = f_{xx} ; \frac{\partial^2 f}{\partial y^2} = f_{yy} ; \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = f_{xy} = f_{yx}$$

partial derivatives of IInd order.

HOMOGENEOUS FUNCTIONS

A function f (x, y, z) is said to be homogeneous in x, y, z of degree n

if $$f(x, y, z) = x^n \phi\left(\frac{y}{x}, \frac{z}{x}\right)$$

e.g. : $f(x,y) = x^3+y^3 = x^3\left(1+\frac{y^3}{x^3}\right) = x^3 \phi\left(\frac{y}{x}\right)$

It is a homogeneous function in x, y of degree 3.

Euler's Theorem : If 'f' be a homogeneous function in x, y and z of degree n

Then $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} = nf$

Composite function : If $z = f(x, y)$ where $x = g(t)$; $y = h(t)$

Volume by Triple Integration :

Volume (Cartesian Coordinates) = $\iiint_V dx\,dy\,dz$

Volume (Cylindrical Coordinates) = $\iiint_V r\,dr\,d\theta\,dz$

Volume (Spherical Coordinates) = $\iiint_V r^2 \sin\theta\,dr\,d\theta\,d\phi$

Volume of solid of revolution :

If the solid is rotated about x-axis

Then volume $V = \iint 2\pi y\,dy\,dx$

If the solid is rotated about y-axis, $V = \iint 2\pi x\,dx\,dy$

SURFACE AND VOLUME ABOUT AXES OF REVOLUTION

Integration is a very significant tool for finding surface and volume of the solid about their axis of revolution.

Volume formulae for catesian equation

Volume of the solid about the x-axis of the curve y = f(x), the x-axis and lines x = a and x = b is given by

$$V = \int_a^b \pi y^2 dx$$

Smularly about y-axis

$$V = \int_a^b \pi x^2 dy$$

for polar coordinates eq. $r = f(\theta)$

we replace $x = r\cos\theta$, $y = r\sin\theta$

Surface formulae for cartesian eq.

Surface of solid about the x-axis of the curve y = f(x), the x-axis and lines x = a, x = b is given by

$$S = \int_a^b 2\pi y\,ds$$

about y-axis

$$S = \int_a^b 2\pi x\,ds$$

Then we put $\frac{\partial F}{\partial x} = 0 = \frac{\partial F}{\partial y} = \frac{\partial F}{\partial z}$ and try to find out Lagranges multipliers and stationary points.

Asymptotes : It is a straight line or surface or curve which is always closer and closer to given curve never touching it. An asymptote | | to x-axis or y-axis is called rectangular asymptote.

— An asymptote | | to x-axis or y-axis can be found out by putting the coefficient of highest powers of x or y is equal to zero.

— An asymptote which is neither | | to x-axis nor | | to y-axis is called oblique asymptote.

MULTIPLE INTEGRALS

This topic covers double and triple integral

If the region R is given by $R = \{(x, y) ; a \le x \le b ; c \le y \le d\}$

$$\int_c^d \int_a^b f(x,y)\,dx\,dy = \int_a^b \int_c^d f(x,y)\,dy\,dx$$

If the region $R = \{(x, y) ; f(x) \le y \le g(x) ; a \le x \le b\}$

$$\text{The given integral} = \int_a^b \int_{f(x)}^{g(x)} f(x,y)\,dy\,dx$$

If the region $R = \{(x, y) ; f_1(y) \le x \le g_1(y) ; c \le y \le d\}$

$$\text{Double Integral becomes} = \int_c^d \int_{f_1(y)}^{g_1(y)} f(x, y)\,dx\,dy$$

If the region $V = \{(x, y, z) ; g_1(y, z) \le x \le g_2(y, z) ; f_1(z) \le y \le f_2(z) ; c \le z \le d\}$

$$\text{Triple Integral} = \int_c^d \int_{f_1(z)}^{f_2(z)} \int_{g_1(y,z)}^{g_2(y,z)} f(x, y, z)\,dx\,dy\,dz$$

Area by Double Integration :

$$\text{Area (Cartesian coordinates)} = \iint_A dx\,dy$$

Where $A = \{(x, y) ; f_1(x) \le y \le f_2(x) ; a \le x \le b\}$

$$\text{Area (polar coordinates)} = \iint_A r\,dr\,d\theta$$

$A = \{(r, \theta) ; \theta_1 \le \theta \le \theta_2 ; f_1(\theta) \le r \le f_2(\theta)\}$

Volume by Double Integration :

$$\text{Volume } V = \iint z\,dx\,dy \text{ or } \iint z r\,dr\,d\theta$$

Then z is said to be composite function of single variable 't'.

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

Similarly if $z = f(x, y)$ where $x = g(u, v)$; $y = h(u, v)$
Then z is said to be composite function of two variables u and v.

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial u}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}$$

and

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}$$

Derivatives of implicit function : If $f(x, y) = c$ be an implicit relation between x and y

Then

$$\frac{dy}{dx} = -\frac{f_x}{f_y}$$

and

$$\frac{d^2y}{dx^2} = -\frac{{f_y}^2 f_{xx} - 2 f_{xy} f_x f_y + {f_x}^2 f_{yy}}{{f_y}^3} \ ; f_y \neq 0$$

Jacobians : If u and v are functions of two independent variables x and y Then Jacobian of u, v with respect to x and y is denoted by $J\left(\frac{u, v}{x, y}\right)$ or $\frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}$

CONDITIONS FOR MAXIMA OR MINIMA

Let the function be $z = f(x, y)$

For maxima or minima we put = $\frac{\partial f}{\partial x} = 0 = \frac{\partial f}{\partial y}$

on solving we get different points (a, b), (c, d), ...

Now we calculate $A = \frac{\partial^2 f}{\partial x^2}$; $B = \frac{\partial^2 f}{\partial x \partial y}$; $C = \frac{\partial^2 f}{\partial y^2}$ at these respective points.

(i) If $AC - B^2 > 0$; $A > 0$ Then the respective point is a point of minima.
(ii) If $AC - B^2 > 0$; $A < 0$ Then the respective point is a point of maxima.
(iii) If $AC - B^2 < 0$ Then the respective point is not a extreme or stationary point.
(iv) If $AC - B^2 = 0$ Then the point is a point of further discussion.

LAGRANGE'S MULTIPLIERS METHOD

Here we form a function called Lagrange's function

$$F(x, y, z) = f(x, y, z) + \lambda_1\phi_1 + \lambda_2\phi_2 + \ldots\ldots$$

Where $f(x, y, z)$ be the function whose maximum or minimum value is to be found out and λ_1, λ_2, ... are Lagrange's multipliers, ϕ_1, ϕ_2, ... are given constraints.

$$b\frac{\partial z}{\partial x} = ab\, e^{ax+by}[f'(ax-by) + f(ax-by)] \quad \text{...(1)}$$

$$\frac{\partial z}{\partial y} = e^{ax+by} f'(ax-by)(-b) + f(ax-by)\, e^{ax+by}.\, b$$

$$\Rightarrow \quad a\frac{\partial z}{\partial y} = e^{ax+by}\, ab[-f'(ax-by) + f(ax-by)] \quad \text{...(2)}$$

adding (1) and (2), we get

$$b\frac{\partial z}{\partial x} + a\frac{\partial z}{\partial y} = ab\, e^{ax+by}[2f(ax-by)] = 2abz.$$

Q 8. Show that the function

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+2y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

is not continuous at (0,0) but its partial derivatives f_x and f_y exist at (0,0).

(PTU, Dec. 2004)

Ans. Let $(x,y) \to (0,0)$ along the curve $y = mx$

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{x\to 0} \frac{mx^2}{x^2+2m^2x^2} = \frac{m}{1+2m^2}$$

which has diff values for diff values of m

Hence the value is not unique

$\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist Hence $f(x,y)$ is not continuous at origin

$$\left(\frac{\partial f}{\partial x}\right)_{(0,0)} = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = \frac{0-0}{h} = 0$$

$$\left(\frac{\partial f}{\partial y}\right)_{(0,0)} = \lim_{k\to 0} \frac{f(0,k)-f(0,0)}{k} = \frac{0-0}{k} = 0$$

Q 9. Find the total derivative of $z = \tan^{-1}\left(\frac{x}{y}\right)$ where $(x, y) \neq (0, 0)$.

(PTU, Dec. 2007)

Ans. Given $z = \tan^{-1}\left(\frac{x}{y}\right)$ ...(1)

Diff (1) partially w.r.t. x, we get

$$\frac{\partial z}{\partial x} = \frac{1}{1+\frac{x^2}{y^2}} \cdot \frac{1}{y} = \frac{y}{x^2+y^2}$$

Diff (1) partially w.r.t. y, we get

$$\frac{\partial z}{\partial y} = \frac{1}{1+\frac{x^2}{y^2}}\left(-\frac{x}{y^2}\right) = \frac{-x}{x^2+y^2}$$

Total derivative $dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = \frac{y}{x^2+y^2}dx - \frac{x}{x^2+y^2}dy$

Q 10. If $u = \sin\left(\frac{x}{y}\right)$ and $x = e^t$, $y = t^2$ find $\frac{du}{dt}$. **(PTU, May 2006)**

Ans. If $u = \sin\left(\frac{x}{y}\right)$, $x = e^t$, $y = t^2$ u is a composite function of t

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} \quad \text{...(1)}$$

$$\frac{\partial u}{\partial x} = \cos\left(\frac{x}{y}\right)\frac{1}{y} \text{ and } \frac{\partial u}{\partial y} = \cos\left(\frac{x}{y}\right)\left(\frac{-x}{y^2}\right)$$

$$\frac{dx}{dt} = e^t, \frac{dy}{dt} = 2t \quad \text{eq (1) gives}$$

$$\frac{du}{dt} = \frac{1}{y}\cos\left(\frac{x}{y}\right)e^t - \frac{x}{y^2}\cos\left(\frac{x}{y^2}\right)2t$$

$$= \frac{e^t}{t^2}\cos\left(\frac{e^t}{t^2}\right) - \frac{e^t}{t^4}\times 2t\cos\left(\frac{e^t}{t^2}\right)$$

$$= \cos\left(\frac{e^t}{t^2}\right)\left[1-\frac{2}{t}\right]\frac{e^t}{t^2}.$$

Q 11. If $x = r\cos\theta$ and $y = r\sin\theta$, find $\frac{\partial(x,y)}{\partial(r,\theta)}$ and $\frac{\partial(r,\theta)}{\partial(x,y)}$.

(PTU, Dec. 2005 ; May 2005)

Ans.

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}; \quad \frac{\partial(r,\theta)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{vmatrix}$$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r \quad \text{...(1)}$$

Q 2. If $u = xy\left(\frac{y}{x}\right)$ then find $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2}$ (PTU, May 2008)

Ans. Given $u = xy\left(\frac{y}{x}\right)$ It is Homogeneous function in x and y of degree 1.

∴ by Euler's theorem, we have

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = u \quad \text{—(1)}$$

Diff. (1) partially w.r.t. x, we get

$$\Rightarrow x\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + y\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial u}{\partial x} \Rightarrow x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x \partial y} = 0 \quad \text{—(2)}$$

Diff. (1) partially w.r.t. y, we get

$$x\frac{\partial^2 u}{\partial x \partial y} + y\frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial y} = \frac{\partial u}{\partial y}$$

$$\Rightarrow x\frac{\partial^2 u}{\partial y \partial x} + y\frac{\partial^2 u}{\partial y^2} = 0 \quad \text{—(3)}$$

Multiply eq (2) by x and eq (3) by y and then adding, we get

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 0$$

Q 3. If $z = \sin^{-1}\left(\frac{\sqrt{x}+\sqrt{y}}{\sqrt{x}+\sqrt{y}}\right)$ then prove that $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 0.$ (PTU, Dec. 2007)

Ans. Given $z = \sin^{-1}\left(\frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}\right) \Rightarrow \sin z = u = \frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}$

i.e. $u = \sin z = \frac{\sqrt{x}}{\sqrt{x}}\left[\frac{1-\sqrt{\frac{y}{x}}}{1+\sqrt{\frac{y}{x}}}\right] \Rightarrow u = x^0\phi\left(\frac{y}{x}\right)$

i.e. u is a homogeneous function of degree 0 in x and y

∴ by Euler's theorem $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 0.u$

$$\Rightarrow x\frac{\partial}{\partial x}(\sin z) + y\frac{\partial}{\partial y}(\sin z) = 0$$

$$\Rightarrow x\cos z\frac{\partial z}{\partial x} + y\cos z\frac{\partial z}{\partial y} = 0 \Rightarrow x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 0$$

Q 4. If u is homogeneous function of degree 'n' in x, y then prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu.$ (PTU, Dec. 2008, 2007, 2004; May 2006, 2005)

Ans. Let z be a Homogeneous function of x and y of degree n

$$\text{i.e.}\quad u = x^n f\left(\frac{y}{x}\right) \quad \text{—(1)}$$

$$\text{and}\quad \frac{\partial u}{\partial x} = x^n f'\left(\frac{y}{x}\right)\left(\frac{-y}{x^2}\right) + f\left(\frac{y}{x}\right) nx^{n-1}$$

$$\frac{\partial u}{\partial y} = x^n f'\left(\frac{y}{x}\right)\frac{1}{x}$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = x\left[-yx^{n-2}f'\left(\frac{y}{x}\right) + nx^{n-1}f\left(\frac{y}{x}\right)\right] + yx^{n-1}f'\left(\frac{y}{x}\right)$$

$$= nx^n f\left(\frac{y}{x}\right) = nu \quad \text{(using (1))}$$

Q 5. Define a homogeneous function with the help of one example. (PTU, Dec. 2009; May 2009, 2005)

Ans. A function of the type $f(x, y) = x^n\phi\left(\frac{y}{x}\right)$ is called homogeneous function in x and y of degree n

e.g. $f(x, y) = x^2 + y^2 + 2xy = x^2\left[1 + \left(\frac{y}{x}\right)^2 + \frac{2y}{x}\right]$

$$= x^2\phi\left(\frac{y}{x}\right)$$

It is a homogeneous function in x and y of degree 2.

Q 6. Define composite function of single and double variables. (PTU, May 2007)

Ans. If $z = \phi(x, y)$

where $x = g(t)$; $y = h(t)$

∴ z is said to be composite function of single variable 't'

further If $z = \phi(u, v)$

where $u = g(x, y)$; $v = h(x, y)$

∴ z is said to be composite function of two variables x and y.

Q 7. If $z = e^{ax+by} f(ax - by)$ find $b\frac{\partial z}{\partial x} + a\frac{\partial z}{\partial y}$. (PTU, May 2006)

Ans. $z = e^{ax+by} f(ax - by)$ —(1)

Diff. eq (1) partially w.r.t. x and y we get

$$\frac{\partial z}{\partial x} = e^{ax+by} f'(ax - by).a + f(ax - by)e^{ax+by}.a$$

where In cartesian coordinates $ds = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$ or $\sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy$

In parametric from $ds = \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\, dt$

In polar form $ds = \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\, d\theta$

QUESTION-ANSWERS

Q 1. If $z = \sin^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$ then prove that $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y} = \frac{1}{2}\tan z$.

(PTU, May 2008)

Ans. Given $z = \sin^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$

$$\Rightarrow \quad \sin z = u = \frac{x+y}{\sqrt{x}+\sqrt{y}}$$

$$= \frac{\left[1+\frac{y}{x}\right]x}{\left[1+\sqrt{\frac{y}{x}}\right]\sqrt{x}} = x^{1/2}\phi\left(\frac{y}{x}\right)$$

$\therefore$ u is homogeneous in x and y with degree $\frac{1}{2}$.

Hence by Euler's theorem, we have

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = \frac{1}{2}u \Rightarrow x\frac{\partial}{\partial x}(\sin z)+y\frac{\partial}{\partial y}(\sin z) = \frac{1}{2}\sin z$$

$$\Rightarrow x\cos z\frac{\partial z}{\partial x}+y\cos z\frac{\partial z}{\partial y} = \frac{1}{2}\sin z$$

$$\Rightarrow \quad x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y} = \frac{1}{2}\tan z.$$

$$= x^{-2}\phi\left(\frac{y}{x}\right)$$

$f(x, y)$ is homo. function of degree -2 in x & y.
By Euler's theorem, we have

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y} = -2f(x,y)$$

or $$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}+2f(x,y)=0$$

Q 20. If $V = \log(x^3+y^3+z^3-3xyz)$ show that $\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 V = \frac{-9}{(x+y+z)^2}$.

(PTU, Dec. 2008; May 2010, 2007)

Ans. Given $V = \log(x^3+y^3+z^3-3xyz)$ ...(1)

Diff. (1) partially w.r.t. x, we get

$$\frac{\partial V}{\partial x} = \frac{1}{x^3+y^3+z^3-3xyz}(3x^2-3yz) \quad ...(2)$$

Diff. (1) partially w.r.t. y, we get

$$\frac{\partial V}{\partial y} = \frac{1}{x^3+y^3+z^3-3xyz}(3y^2-3xz) \quad ...(3)$$

Similarly Diff. (1) partially w.r.t. z, we get

$$\frac{\partial V}{\partial z} = \frac{1}{x^3+y^3+z^3-3xyz}(3z^2-3xy) \quad ...(4)$$

$$\frac{\partial V}{\partial x}+\frac{\partial V}{\partial y}+\frac{\partial V}{\partial z} = \frac{3(x^2+y^2+z^2-xy-yz-zx)}{x^3+y^3+z^3-3xyz}$$

$$= \frac{3(x^2+y^2+z^2-xy-yz-zx)}{(x+y+z)(x^2+y^2+z^2-xy-yz-zx)} = \frac{3}{x+y+z}$$

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 V = \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial V}{\partial x}+\frac{\partial V}{\partial y}+\frac{\partial V}{\partial z}\right)$$

$$= \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{3}{x+y+z}\right)$$

$$= \frac{-3}{(x+y+z)^2} - \frac{3}{(x+y+z)^2} - \frac{3}{(x+y+z)^2}$$

$$= \frac{-9}{(x+y+z)^2}$$

Q 21. If $u = r^m$ and $r^2 = x^2+y^2+z^2$,

Show that $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2} = m(m+1)r^{m-2}$. (PTU, Dec. 2006)

Ans.
$$u = r^m \quad ...(1)$$
$$r^2 = x^2+y^2+z^2 \quad ...(2)$$

Diff. (1) partially w.r.t. x

$$\frac{\partial u}{\partial x} = mr^{m-1}\frac{\partial r}{\partial x} \quad \text{also (2) gives } \frac{\partial r}{\partial x} = \frac{x}{r}$$

$$= mr^{m-1}\frac{x}{r} = mr^{m-2}x$$

$$\frac{\partial^2 u}{\partial x^2} = m\left[r^{m-2} + x(m-2)r^{m-3}\frac{\partial r}{\partial x}\right] = m\left[r^{m-2} + x^2(m-2)r^{m-4}\right]$$

$$\frac{\partial^2 u}{\partial y^2} = m\left[r^{m-2} + (m-2)r^{m-4}y^2\right]$$

Similarly $\frac{\partial^2 u}{\partial z^2} = m\left[r^{m-2} + (m-2)r^{m-4}z^2\right]$ on adding, we get

$$u_{xx}+u_{yy}+u_{zz} = m\left[3r^{m-2} + (m-2)r^{m-4}(x^2+y^2+z^2)\right]$$
$$= mr^{m-2}[3+m-2]$$
$$= m(m+1)r^{m-2}$$

Q 22. If $u = \sin^{-1}(x-y)$, $x = 3t$, $y = 4t^3$, find the value of $\frac{du}{dt}$. (PTU, Dec. 2005)

Ans. $u = \sin^{-1}(x-y)$, $x = 3t$, $y = 4t^3$

u is a composite function of t

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} \quad ...(1)$$

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{1-(x-y)^2}}, \quad \frac{\partial u}{\partial y} = \frac{-1}{\sqrt{1-(x-y)^2}}$$

$$\frac{dx}{dt} = 3, \quad \frac{dy}{dt} = 12t^2 \quad \text{eq (1) gives}$$

$$\frac{du}{dt} = \frac{3}{\sqrt{1-(x-y)^2}} - \frac{1}{\sqrt{1-(x-y)^2}}\,12t^2 = \frac{3(1-4t^2)}{\sqrt{1-(3t-4t^3)^2}}$$

$$\frac{du}{dt} = \frac{3(1-4t^2)}{\sqrt{1-9t^2-16t^6+24t^4}} = \frac{3(1-4t^2)}{\sqrt{(1-t^2)(1-8t^2+16t^4)}}$$

Q 16. If $u = \sin^{-1}\left(\dfrac{x^3+y^3+z^3}{ax+by+cz}\right)$. Then prove that $xu_x + yu_y + zu_z = 2\tan u$.

(PTU, May 2008 ; June 2007)

Ans. Given $u = \sin^{-1}\left(\dfrac{x^3+y^3+z^3}{ax+by+cz}\right) \Rightarrow \sin u = \dfrac{x^3\left[1+\left(\frac{y}{x}\right)^3+\left(\frac{z}{x}\right)^3\right]}{x\left[a+\frac{by}{x}+\frac{cz}{x}\right]}$

i.e. $\sin u = x^2 f\left(\frac{y}{x}, \frac{z}{x}\right)$

$\therefore$ sin u is a homogeneous function of degree 2 in x, y, z.

$\therefore$ by Euler's theorem

$$\Rightarrow x\frac{\partial}{\partial x}(\sin u) + y\frac{\partial}{\partial y}(\sin u) + z\frac{\partial}{\partial z}(\sin u) = 2\sin u$$

$$\Rightarrow x\cos u\frac{\partial u}{\partial x} + y\cos u\frac{\partial u}{\partial y} + z\cos u\frac{\partial u}{\partial z} = 2\sin u$$

$$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 2\tan u$$

Q 17. If $u = \tan^{-1}\left(\dfrac{y^2}{x}\right)$. Prove that $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial y\partial x} + y^2\dfrac{\partial^2 u}{\partial y^2} = -\sin^2 u \sin 2u$.

(PTU, May 2011, 2008)

Ans. tan u is a homo. function of degree 1 in y & x $\left(\because \tan u = x\left(\frac{y}{x}\right)^2\right)$

$$x\frac{\partial}{\partial x}(\tan u) + y\frac{\partial}{\partial y}(\tan u) = \tan u \quad \text{[using Euler's theorem]}$$

or $$x\sec^2 u\frac{\partial u}{\partial x} + y\sec^2 u\frac{\partial u}{\partial y} = \tan u$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}\sin 2u \quad ...(1)$$

Diff (1) partially w.r.t. x

$$x\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + y\frac{\partial^2 u}{\partial x\partial y} = \cos 2u\frac{\partial u}{\partial x} \quad ...(2)$$

Diff (1) partially w.r.t. y

$$x\frac{\partial^2 u}{\partial x\partial y} + y\frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial y} = \cos 2u\frac{\partial u}{\partial y} \quad ...(3)$$

eq (2) × x + eq (3) × y

$$\Rightarrow x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} + \left[x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right] = \cos 2u\left[x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right]$$

$$\Rightarrow x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} + \frac{1}{2}\sin 2u = \cos 2u\,\frac{1}{2}\sin 2u \quad \text{(using (1))}$$

$$\Rightarrow x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \frac{-1}{2}\sin 2u\,(1-\cos 2u)$$

$$= \frac{-1}{2}\sin 2u \cdot 2\sin^2 u$$

$$= -\sin 2u\sin^2 u$$

Q 18. Verify Euler's theorem for $f(x, y, z) = 3x^2yz + 5xy^2z + 4z^4$. (PTU, Dec. 2008)

Ans. $f(x, y, z) = 3x^2yz + 5xy^2z + 4z^4$

$$= x^4\left[3\left(\frac{y}{x}\right)\left(\frac{z}{x}\right) + 5\left(\frac{y}{x}\right)^2\left(\frac{z}{x}\right) + 4\left(\frac{z}{x}\right)^4\right]$$

$$= x^4\phi\left(\frac{y}{x}, \frac{z}{x}\right)$$

i.e. f(x, y, z) be a homogeneous function in x, y, z of degree 4 so for verification of Euler's theorem. We have to verify that $x\dfrac{\partial f}{\partial x} + y\dfrac{\partial f}{\partial y} + z\dfrac{\partial f}{\partial z} = 4f$

Now $\dfrac{\partial f}{\partial x} = 6xyz + 5y^2z$

$\dfrac{\partial f}{\partial y} = 3x^2z + 10xyz$

$\dfrac{\partial f}{\partial z} = 3x^2y + 5xy^2 + 16z^3$

Now $x\dfrac{\partial f}{\partial x} + y\dfrac{\partial f}{\partial y} + z\dfrac{\partial f}{\partial z} = 6x^2yz + 5xy^2z + 3x^2yz + 10xy^2z + 3x^2yz + 5xy^2z + 16z^4$

$= 12x^2yz + 20xy^2z + 16z^4$

$= 4(3x^2yz + 5xy^2z + 4z^4) = 4f$

$\therefore$ Euler's theorem verified.

Q 19. If $f(x, y) = \dfrac{1}{x^2} + \dfrac{1}{xy} + \dfrac{\log x - \log y}{x^2 + y^2}$ then show that $x\dfrac{\partial f}{\partial x} + y\dfrac{\partial f}{\partial y} = -2f$.

(PTU, May 2004)

Ans. $$f(x, y) = \frac{1}{x^2}\left[1 + \frac{1}{\frac{y}{x}} - \frac{\log\frac{y}{x}}{1+\left(\frac{y}{x}\right)^2}\right]$$

Now $x = r\cos\theta,\ y = r\sin\theta$

$$\sqrt{x^2+y^2} = r \ \&\ \tan\theta = \frac{y}{x}$$

$$\theta = \tan^{-1}\frac{y}{x}$$

$$\frac{\partial(r,\theta)}{\partial(x,y)} = \begin{vmatrix} \frac{2x}{2\sqrt{x^2+y^2}} & \frac{2y}{2\sqrt{x^2+y^2}} \\ \frac{1}{1+\frac{y^2}{x^2}}\left(\frac{-y}{x^2}\right) & \frac{1}{1+\frac{y^2}{x^2}}\cdot\frac{1}{x} \end{vmatrix} = \begin{vmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{vmatrix}$$

$$= \frac{x^2}{(x^2+y^2)^{\frac{3}{2}}} + \frac{y^2}{(x^2+y^2)^{\frac{3}{2}}} = \frac{x^2+y^2}{(x^2+y^2)^{\frac{3}{2}}}$$

$$= \left(\sqrt{x^2+y^2}\right)^{-1}$$

$$= 1/r \qquad ...(2)$$

From (1) & (2), we get the result.

Q 12. State Euler's theorem and use it to prove that $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2}$ = 0, whenever $u = x\phi\left(\frac{y}{x}\right) + \psi\left(\frac{y}{x}\right)$. (PTU, May 2012)

Solution. If f be a homogeneous function in x and y of degree n

Then, $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf$ ...(1)

Given, $u = x\phi\left(\frac{y}{x}\right) + \psi\left(\frac{y}{x}\right) = v + w$

where $v = x\phi\left(\frac{y}{x}\right)$ and $w = \psi\left(\frac{y}{x}\right)$

Now v is a homogeneous function in x and y of degree 1 and w is a homogeneous function of degree 0.

Therefore by Euler's theorem

$$x^2\frac{\partial^2 v}{\partial x^2} + 2xy\frac{\partial^2 v}{\partial x \partial y} + y^2\frac{\partial^2 v}{\partial y^2} = 0 \qquad ...(2)\ [\because n = 1]$$

Similarly $$x^2\frac{\partial^2 w}{\partial x^2} + 2xy\frac{\partial^2 w}{\partial x \partial y} + y^2\frac{\partial^2 w}{\partial y^2} = 0 \qquad ...(3)\ [\because n = 0]$$

[As if z is a homogeneous function in x and y of degree n then by Euler's theorem

$$x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x \partial y} + y^2\frac{\partial^2 z}{\partial y^2} = n(n-1)z]$$

Adding (2) and (3), we have

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 0\ [\because u = v + w].$$

Q 13. If $u = x^2 - 2y$ and $v = x + y$. Find $\frac{\partial(u,v)}{\partial(x,y)}$. (PTU, Dec. 2006)

Ans. $\frac{\partial u}{\partial x} = 2x,\ \frac{\partial u}{\partial y} = -2$

$\frac{\partial v}{\partial x} = 1,\ \frac{\partial v}{\partial y} = 1$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 2x & -2 \\ 1 & 1 \end{vmatrix} = 2x + 2$$

Q 14. If $u = x\sin y$ and $v = y\sin x$ then find $\frac{\partial(u,v)}{\partial(x,y)}$. (PTU, May 2006)

Ans. Given $u = x\sin y$, $v = y\sin x$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \sin y & x\cos y \\ y\cos x & \sin x \end{vmatrix}$$

$$= \sin x \sin y - xy\cos x\cos y$$

Q 15. If $u = x^3 + xy$ and $v = xy$. Find $\frac{\partial(u,v)}{\partial(x,y)}$. (PTU, May 2009)

Ans. Given $u = x^3 + xy$; $v = xy$

$$\frac{\partial u}{\partial x} = 3x^2 + y;\ \frac{\partial u}{\partial y} = x;\ \frac{\partial v}{\partial x} = y;\ \frac{\partial v}{\partial y} = x$$

Now $$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 3x^2+y & x \\ y & x \end{vmatrix} = 3x^3 + xy - xy = 3x^3$$

$$= \sin^2\theta \frac{\partial^2 u}{\partial r^2} + \frac{2\sin\theta\cos\theta}{r}\frac{\partial^2 u}{\partial r \partial\theta} + \frac{\cos^2\theta}{r^2}\frac{\partial^2 u}{\partial\theta^2}$$

$$+ \frac{\cos^2\theta}{r}\frac{\partial u}{\partial r} - \frac{2\cos\theta\sin\theta}{r^2}\frac{\partial u}{\partial\theta} \quad \text{—(2)} \qquad \left[\because \frac{\partial^2 u}{\partial r\partial\theta} = \frac{\partial^2 u}{\partial\theta\partial r}\right]$$

on adding (1) and (2), we get

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \left(\sin^2\theta + \cos^2\theta\right)\frac{\partial^2 u}{\partial r^2} + \frac{\left(\sin^2\theta + \cos^2\theta\right)}{r^2}\frac{\partial^2 u}{\partial\theta^2} + \frac{\left(\sin^2\theta + \cos^2\theta\right)}{r}\frac{\partial u}{\partial r}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2} + \frac{1}{r}\frac{\partial u}{\partial r}$$

i.e. $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$ transform to $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}$ in polar coordinates.

Q 28. If $z = f(x, y)$, $x = r\cos\theta$, $y = r\sin\theta$, then show that

$$\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 = \left(\frac{\partial f}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial f}{\partial\theta}\right)^2.$$ **(PTU, Dec. 2007)**

Ans. $x = r\cos\theta,\ y = r\sin\theta$ —(1)

Squaring and adding, we get

$x^2 + y^2 = r^2$ —(2)

on dividing, $\tan\theta = \frac{y}{x} \Rightarrow \theta = \tan^{-1}\frac{y}{x}$

$$2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r} = \frac{r\cos\theta}{r} = \cos\theta$$

also $\frac{\partial r}{\partial y} = \frac{y}{r} = \sin\theta$

$$\frac{\partial\theta}{\partial x} = \frac{1}{1+\frac{y^2}{x^2}} \times \left(\frac{-y}{x^2}\right) = \frac{-y}{x^2+y^2} = \frac{-r\sin\theta}{r^2} = \frac{-\sin\theta}{r}$$

$$\frac{\partial\theta}{\partial y} = \frac{1}{1+\frac{y^2}{x^2}}\left(\frac{1}{x}\right) = \frac{x}{x^2+y^2} = \frac{r\sin\theta}{r^2} = \frac{\cos\theta}{r}$$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial f}{\partial\theta}\frac{\partial\theta}{\partial x}$$

$$= \frac{\partial f}{\partial r}\cos\theta + \frac{\partial f}{\partial\theta}\frac{(-\sin\theta)}{r} \quad \text{—(1)}$$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial f}{\partial\theta}\frac{\partial\theta}{\partial y}$$

$$= \frac{\partial f}{\partial r}\sin\theta + \frac{\partial f}{\partial\theta}\frac{\cos\theta}{r} \quad \text{—(2)}$$

on squaring & adding (1) & (2)

$$\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 = \left(\frac{\partial f}{\partial r}\right)^2(\sin^2\theta + \cos^2\theta) + \left(\frac{\partial f}{\partial\theta}\right)^2\frac{1}{r^2}(\sin^2\theta + \cos^2\theta)$$

$$= \left(\frac{\partial f}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial f}{\partial\theta}\right)^2$$

Q 29. If $u = f(y - z, z - x, x - y)$, prove that : $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$. **(PTU, Dec. 2008)**

Ans. Given $u = f(y - z, z - x, x - y)$

i.e. $u = f(X, Y, Z)$ where $X = y - z$; $Y = z - x$; $Z = x - y$

u is a composite function x, y and z.

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial X}\frac{\partial X}{\partial x} + \frac{\partial u}{\partial Y}\frac{\partial Y}{\partial x} + \frac{\partial u}{\partial Z}\frac{\partial Z}{\partial x}$$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial u}{\partial X}(0) + \frac{\partial u}{\partial Y}(-1) + \frac{\partial u}{\partial Z}(1) = -\frac{\partial u}{\partial Y} + \frac{\partial u}{\partial Z}(1) \quad \text{—(1)}$$

Now $\frac{\partial u}{\partial y} = \frac{\partial u}{\partial X}\frac{\partial X}{\partial y} + \frac{\partial u}{\partial Y}\frac{\partial Y}{\partial y} + \frac{\partial u}{\partial Z}\frac{\partial Z}{\partial y}$

$$\Rightarrow \frac{\partial u}{\partial y} = \frac{\partial u}{\partial X}\,1 + \frac{\partial u}{\partial Y}(0) + \frac{\partial u}{\partial Z}(-1) = \frac{\partial u}{\partial X} - \frac{\partial u}{\partial Z} \quad \text{—(2)}$$

again $\frac{\partial u}{\partial z} = \frac{\partial u}{\partial X}\frac{\partial X}{\partial z} + \frac{\partial u}{\partial Y}\frac{\partial Y}{\partial z} + \frac{\partial u}{\partial Z}\frac{\partial Z}{\partial z}$

$$= \frac{\partial u}{\partial X}(-1) + \frac{\partial u}{\partial Y}(1) + \frac{\partial u}{\partial Z}(0) = \frac{-\partial u}{\partial X} + \frac{\partial u}{\partial Y} \quad \text{—(3)}$$

on adding (1), (2) and (3), we get

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0.$$

Q 30. If ρ_1 and ρ_2 be the radii of curvature at the ends of a focal chord of the parabola $y^2 = 4ax$ then show that, $\rho_1^{-2/3} + \rho_2^{-2/3} = (2a)^{-2/3}$. **(PTU, Dec. 2008)**

Ans. The eq. of given parabola be $y^2 = 4ax$

$$\frac{dy}{dx} = \frac{2a}{y};\ \frac{d^2y}{dx^2} = \frac{-2a}{y^2}\frac{dy}{dx} = \frac{-4a^2}{y^3}$$

Q 26. If u = f(x, y, z) and $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$ then show that:

$$\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2=\left(\frac{\partial u}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2+\frac{1}{r^2\sin^2\theta}\left(\frac{\partial u}{\partial \phi}\right)^2$$ **(PTU, Dec. 2004)**

Ans. $u = f(x, y, z)$; $x = r\sin\theta\cos\phi$; $y = r\sin\theta\sin\phi$; $z = r\cos\theta$

u is a composite function of r, θ, ϕ

$$\frac{\partial u}{\partial r}=\frac{\partial u}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial r}+\frac{\partial u}{\partial z}\frac{\partial z}{\partial r}$$

$$\frac{\partial u}{\partial r}=\sin\theta\cos\phi\frac{\partial u}{\partial x}+\sin\theta\sin\phi\frac{\partial u}{\partial y}+\cos\theta\frac{\partial u}{\partial z} \quad ...(1)$$

$$\frac{\partial u}{\partial \theta}=\frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta}+\frac{\partial u}{\partial z}\frac{\partial z}{\partial \theta}$$

$$=r\cos\theta\cos\phi\frac{\partial u}{\partial x}+r\cos\theta\sin\phi\frac{\partial u}{\partial y}-r\sin\theta\frac{\partial u}{\partial z}$$

$$\frac{1}{r}\frac{\partial u}{\partial \theta}=\cos\theta\cos\phi\frac{\partial u}{\partial x}+\cos\theta\sin\phi\frac{\partial u}{\partial y}-\sin\theta\frac{\partial u}{\partial z} \quad ...(2)$$

$$\frac{\partial u}{\partial \phi}=\frac{\partial u}{\partial x}\frac{\partial x}{\partial \phi}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial \phi}+\frac{\partial u}{\partial z}\frac{\partial z}{\partial \phi}$$

$$=-r\sin\theta\sin\phi\frac{\partial u}{\partial x}+r\sin\theta\cos\phi\frac{\partial u}{\partial y}$$

$$\frac{1}{r\sin\theta}\frac{\partial u}{\partial \phi}=-\sin\phi\frac{\partial u}{\partial x}+\cos\phi\frac{\partial u}{\partial y} \quad ...(3)$$

squaring and adding (1) & (2), we get

$$\left(\frac{\partial u}{\partial r}\right)^2+\left(\frac{1}{r}\frac{\partial u}{\partial \theta}\right)^2=\cos^2\phi\left(\frac{\partial u}{\partial x}\right)^2+\sin^2\phi\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2+2\cos\phi\sin\phi\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} \quad ...(4)$$

squaring (3), we get

$$\left(\frac{1}{r\sin\theta}\frac{\partial u}{\partial \phi}\right)^2=\sin^2\phi\left(\frac{\partial u}{\partial x}\right)^2+\cos^2\phi\left(\frac{\partial u}{\partial y}\right)^2-2\sin\phi\cos\phi\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} \quad ...(5)$$

adding (4) & (5) we get

$$\left(\frac{\partial u}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2+\frac{1}{r^2\sin^2\theta}\left(\frac{\partial u}{\partial \phi}\right)^2=\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2$$

Q 27. Transform $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}$ to polar co-ordinates. **(PTU, May 2010; Dec. 2007)**

Ans. Changing to polar coordinates by the transformation

$$x = r\cos\theta,\ y = r\sin\theta$$

⇒ squaring and adding $x^2+y^2=r^2$

and on dividing, $\tan\theta=\frac{y}{x} \Rightarrow \theta=\tan^{-1}\frac{y}{x}$

$$\frac{\partial r}{\partial x}=\frac{x}{r}=\frac{r\cos\theta}{r}=\cos\theta$$

$$\frac{\partial r}{\partial y}=\frac{y}{r}=\frac{r\sin\theta}{r}=\sin\theta$$

$$\frac{\partial \theta}{\partial x}=\frac{1}{1+\frac{y^2}{x^2}}\times\left(\frac{-y}{x^2}\right)=\frac{-y}{x^2+y^2}=\frac{-r\sin\theta}{r^2}=\frac{-\sin\theta}{r}$$

$$\frac{\partial \theta}{\partial y}=\frac{1}{1+\frac{y^2}{x^2}}\left(\frac{1}{x}\right)=\frac{x}{x^2+y^2}=\frac{r\cos\theta}{r^2}=\frac{\cos\theta}{r}$$

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial r}\frac{\partial r}{\partial x}+\frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x}=\cos\theta\frac{\partial u}{\partial r}-\frac{\sin\theta}{r}\frac{\partial u}{\partial \theta}$$

⇒ $$\frac{\partial}{\partial x}=\left(\cos\theta\frac{\partial}{\partial r}-\frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right)$$

and $$\frac{\partial^2 u}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)=\left(\cos\theta\frac{\partial}{\partial r}-\frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right)\left(\cos\theta\frac{\partial u}{\partial r}-\frac{\sin\theta}{r}\frac{\partial u}{\partial \theta}\right)$$

$$=\cos^2\theta\frac{\partial^2 u}{\partial r^2}-\cos\theta\sin\theta\left[\frac{-1}{r^2}\frac{\partial u}{\partial \theta}+\frac{1}{r}\frac{\partial^2 u}{\partial r\partial \theta}\right]$$

$$-\frac{\sin\theta}{r}\left[-\sin\theta\frac{\partial u}{\partial r}+\cos\theta\frac{\partial^2 u}{\partial \theta\partial r}\right]+\frac{\sin\theta}{r^2}\left[\cos\theta\frac{\partial u}{\partial \theta}+\sin\theta\frac{\partial^2 u}{\partial \theta^2}\right]$$

$$=\cos^2\theta\frac{\partial^2 u}{\partial r^2}-\frac{2\cos\theta\sin\theta}{r}\frac{\partial^2 u}{\partial \theta\partial r}+\frac{\sin^2\theta}{r^2}\frac{\partial^2 u}{\partial \theta^2}+\frac{\sin^2\theta}{r}\frac{\partial u}{\partial r}+\frac{2\sin\theta\cos\theta}{r^2}\frac{\partial u}{\partial \theta} \quad ...(1)$$

Again $$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial r}\frac{\partial r}{\partial y}+\frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial y}=\sin\theta\frac{\partial u}{\partial r}+\frac{\cos\theta}{r}\frac{\partial u}{\partial \theta}$$

⇒ $$\frac{\partial}{\partial y}=\sin\theta\frac{\partial}{\partial r}+\frac{\cos\theta}{r}\frac{\partial}{\partial \theta}$$

∴ $$\frac{\partial^2 u}{\partial y^2}=\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)=\left(\sin\theta\frac{\partial}{\partial r}+\frac{\cos\theta}{r}\frac{\partial}{\partial \theta}\right)\left(\sin\theta\frac{\partial u}{\partial r}+\frac{\cos\theta}{r}\frac{\partial u}{\partial \theta}\right)$$

$$=\sin^2\theta\frac{\partial^2 u}{\partial r^2}+\sin\theta\cos\theta\left[\frac{-1}{r^2}\frac{\partial u}{\partial \theta}+\frac{1}{r}\frac{\partial^2 u}{\partial r\partial \theta}\right]$$

$$+\frac{\cos\theta}{r}\left[\cos\theta\frac{\partial u}{\partial r}+\sin\theta\frac{\partial^2 u}{\partial \theta\partial r}\right]+\frac{\cos\theta}{r^2}\left[-\sin\theta\frac{\partial u}{\partial \theta}+\cos\theta\frac{\partial^2 u}{\partial \theta^2}\right]$$

$$= \frac{3(1-4t^2)}{\sqrt{(1-t^2)(1-4t^2)^2}} = \frac{3}{\sqrt{1-t^2}}$$

Q 23. If $u = \frac{x+y}{1-xy}$ and $v = \tan^{-1}x + \tan^{-1}y$, then find the value of $\frac{\partial(u,v)}{\partial(x,y)}$. **(PTU, May 2012)**

Solution. Given $u = \frac{x+y}{1-xy}$ and $v = \tan^{-1}x + \tan^{-1}y$

$$\frac{\partial u}{\partial x} = \frac{(1-xy)-(x+y)(-y)}{(1-xy^2)} = \frac{1+y^2}{(1-xy)^2};\ \frac{\partial v}{\partial x} = \frac{1}{1+x^2}$$

$$\frac{\partial u}{\partial y} = \frac{(1-xy)+(x+y)x}{(1-xy)^2} = \frac{1+x^2}{(1-xy)^2};\ \frac{\partial v}{\partial y} = \frac{1}{1+y^2}$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{1+y^2}{(1-xy)^2} & \frac{1+x^2}{(1-xy)^2} \\ \frac{1}{1+x^2} & \frac{1}{1+y^2} \end{vmatrix}$$

$$= \frac{1}{(1-xy)^2} - \frac{1}{(1-xy)^2} = 0$$

Q 24. If $\frac{x^2}{a^2+u} + \frac{y^2}{b^2+u} + \frac{z^2}{c^2+u} = 1$,

show that : $\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 = 2\left[x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}\right]$. **(PTU, Dec. 2003)**

Ans. Given $x^2(a^2+u)^{-1} + y^2(b^2+u)^{-1} + z^2(c^2+u)^{-1} = 1$ ...(1)

Diff (1) partially w.r.t. x, we get

$$(a^2+u)^{-1}2x + (-1)(a^2+u)^{-2}x^2\frac{\partial u}{\partial x} - y^2(b^2+u)^{-2}\frac{\partial u}{\partial x} - z^2(c^2+u)^{-2}\frac{\partial u}{\partial x} = 0$$

$$\Rightarrow \frac{2x}{a^2+u} = \left[\frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} + \frac{z^2}{(c^2+u^2)}\right]\frac{\partial u}{\partial x}$$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{2x}{(a^2+u)V} \text{ where } V = \frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} + \frac{z^2}{(c^2+u)^2}$$

$$\frac{\partial u}{\partial y} = \frac{2y}{(b^2+u)V} \text{ and } \frac{\partial u}{\partial z} = \frac{2z}{(c^2+u)V}$$

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 = \frac{4}{V^2}\left[\frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} + \frac{z^2}{(c^2+u)^2}\right]$$

$$= \frac{4}{V^2} \cdot V = \frac{4}{V} \quad \text{...(2)}$$

$$\text{Now } 2\left[x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}\right] = 2\left[\frac{2x^2}{(a^2+u)V} + \frac{2y^2}{(b^2+u)V} + \frac{2z^2}{(c^2+u)V}\right]$$

$$= \frac{4}{V} \cdot 1 \text{ (using eq (1))} \quad \text{...(3)}$$

From (2) and (3) we get

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 = 2\left[x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}\right].$$

Q 25. If z is a function of x and y, and u, v are two other variables such that $u = lx + my$, $v = ly - mx$, then show that :

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = (l^2+m^2)\left(\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2}\right).$$ **(PTU, May 2004)**

Ans. Now, z is a composite function of x, y

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}$$

$$= l\frac{\partial z}{\partial u} - m\frac{\partial z}{\partial v} \Rightarrow \frac{\partial}{\partial x} = l\frac{\partial}{\partial u} - m\frac{\partial}{\partial v}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2} = \left(l\frac{\partial}{\partial u} - m\frac{\partial}{\partial v}\right)\left(l\frac{\partial z}{\partial u} - m\frac{\partial z}{\partial v}\right)$$

$$= l^2\frac{\partial^2 z}{\partial u^2} - 2lm\frac{\partial^2 z}{\partial u\partial v} + m^2\frac{\partial^2 z}{\partial v^2} \quad \text{...(1)} \left(\because \frac{\partial^2 z}{\partial u\partial v} = \frac{\partial^2 z}{\partial v\partial u}\right)$$

Similarly, $$\frac{\partial^2 z}{\partial y^2} = m^2\frac{\partial^2 z}{\partial u^2} + 2lm\frac{\partial^2 z}{\partial v\partial u} + l^2\frac{\partial^2 z}{\partial v^2} \quad \text{...(2)}$$

From (1) & (2) and on adding, we get

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = (l^2+m^2)\left(\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2}\right)$$

Thus partial derivative f_x represents the slope of the tangent to the curve $z = f(x, y), y = b$ and partial derivative f_y represents the slope of the tangent to the curve $z = f(x, y), x = a$.

Q 39. If $u(x, y) = x^y$, find $\frac{\partial^2 u}{\partial x \partial y}$ at (1, 2). (PTU, May 2011)

Solution. $u(x, y) = x^y$ ...(1)

Diff. (1) partially w.r.t y, we have

$$\frac{\partial u}{\partial y} = x^y \log x$$

$$\frac{\partial^2 u}{\partial x \partial y} = x^y \cdot \frac{1}{x} + \log x \cdot y x^{y-1}$$

$$\left(\frac{\partial^2 u}{\partial x \partial y}\right)_{(1,2)} = 1^2 \cdot \frac{1}{1} + \log 1 \cdot 2 \cdot 1^{2-1} = 1 + 0 = 1.$$

Q 40. If $\theta = t^n e^{-r^2/4t}$, what value of 'n' will make $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \theta}{\partial r}\right) = \frac{\partial \theta}{\partial t}$. (PTU, Dec. 2010)

Solution. Given $\theta = t^n e^{-r^2/4t}$ ...(1)

Diff. eqn (1) partially w.r.t 't', we get

$$\frac{\partial \theta}{\partial t} = t^n e^{-r^2/4t}\left(+\frac{r^2}{4t^2}\right) + nt^{n-1} e^{-r^2/4t}$$ ...(2)

Diff. eqn (1) partially w.r.t 'r'

$$\frac{\partial \theta}{\partial r} = t^n e^{-r^2/4t}\left(-\frac{r}{2t}\right) = -\frac{t^{n-1}}{2} r e^{-\frac{r^2}{4t}}$$

i.e.

$$r^2 \frac{\partial \theta}{\partial r} = -\frac{1}{2} t^{n-1} r^3 e^{-r^2/4t}$$

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial \theta}{\partial r}\right) = -\frac{1}{2} t^{n-1}\left[r^3 e^{-r^2/4t}\left(-\frac{r}{2t}\right) + e^{-r^2/4t}\, 3r^2\right]$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \theta}{\partial r}\right) = -\frac{1}{2} t^{n-1} e^{-r^2/4t}\left[-\frac{r^2}{2t} + 3\right]$$ ...(3)

Now, $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \theta}{\partial r}\right) = \frac{\partial \theta}{\partial t}$

Thus, $\frac{r^2}{4} t^{n-2} e^{-r^2/4t} - \frac{3}{2} t^{n-1} e^{-r^2/4t} = \frac{1}{4} t^{n-2} r^2 e^{-r^2/4t} + nt^{n-1} e^{-r^2/4t}$

$\Rightarrow$ $$n = -\frac{3}{2}$$

Q 41. State the method to find maxima and minima of $z = f(x, y)$ using partial derivatives. (PTU, Dec. 2006; May 2006)

OR

Discuss the extreme values of $z = f(x, y)$. (PTU, Dec. 2006)

Solution. For maxima or minima we have to put $\frac{\partial f}{\partial x} = 0 = \frac{\partial f}{\partial y}$

Then we find out A, B, C

Where $A = f_{xx}$; $B = f_{xy}$; $C = f_{yy}$ at these respective points

Evaluate $AC - B^2$

(i) If $AC - B^2 > 0$; $A > 0$ Then the given point is a point of minima.

(ii) If $AC - B^2 > 0$; $A < 0$ Then we have point of maxima.

(iii) If $AC - B^2 < 0$ Then the said point is not an extreme point.

(iv) If $AC - B^2 = 0$ Then the said point is a point of further investigation.

Q 42. Explain the Lagrange's method of multipliers for maxima and minima. (PTU, Dec. 2007)

Solution. Let us define a function $F = f + \lambda_1 \phi_1 + \lambda_2 \phi_2 + \ldots$

Where f be the function whose maximum and minimum values we have to found out and $\lambda_1, \lambda_2, \lambda_3, \ldots$ are constants called lagrange's multipliers independent of $x_1, x_2, \ldots$ and $\phi_1, \phi_2, \ldots$ are the given constraints.

Then find out $\frac{\partial F}{\partial x_1}, \frac{\partial F}{\partial x_2}, \ldots$ and equating to 0 and then solve for $x_1, x_2, \ldots$

This method can be applied to those problems which contains three variables and two or more given constraints.

Q 43. Find the extreme value of : $x^2 + y^2 + 6x + 12$. (PTU, Dec. 2005)

Solution. Given $f(x, y) = x^2 + y^2 + 6x + 12$

For maxima or minima put $\frac{\partial f}{\partial x} = 0 = \frac{\partial f}{\partial y}$

$$\frac{\partial f}{\partial x} = 2x + 6 = 0;\ \frac{\partial f}{\partial y} = 2y = 0$$

$\Rightarrow$ $$x = -3,\ y = 0$$

$$\frac{\partial^2 f}{\partial x^2} = 2;\ \frac{\partial^2 f}{\partial x \partial y} = 0;\ \frac{\partial^2 f}{\partial y^2} = 2$$

at (−3, 0)

$$A = \frac{\partial^2 f}{\partial x^2} = 2;\ B = 0;\ C = 2 \quad AC - B^2 = 4 > 0;\ A = 2 > 0$$

(−3, 0) is a point of minima and min value = 9 − 18 + 12 = 3

Q 44. Find the point on the surface of $z = x^2 + y^2 + 10$ nearest to the plane $x + 2y - z = 0$. (PTU, May 2005)

Solution. The given surface $z = x^2 + y^2 + 10$

$$\frac{\partial(r,\theta)}{\partial(x,y)} = \begin{vmatrix} \frac{2x}{2\sqrt{x^2+y^2}} & \frac{2y}{2\sqrt{x^2+y^2}} \\ \frac{1}{1+\frac{y^2}{x^2}}\left(\frac{-y}{x^2}\right) & \frac{1}{1+\frac{y^2}{x^2}}\cdot\frac{1}{x} \end{vmatrix} = \begin{vmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{vmatrix}$$

$$= \frac{x^2}{(x^2+y^2)^{\frac{3}{2}}} + \frac{y^2}{(x^2+y^2)^{\frac{3}{2}}} = \frac{x^2+y^2}{(x^2+y^2)^{\frac{3}{2}}}$$

$$= \left(\sqrt{x^2+y^2}\right)^{-1}$$

$$= 1/r$$

Q 34. Use Euler's theorem to show that

$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2u \log u$, where $u = e^{x^2+y^2}$. (PTU, May 2010)

Solution. Given $u = e^{x^2+y^2} \Rightarrow \log u = x^2+y^2 = x^2\left(1+\frac{y^2}{x^2}\right) = x^2\phi\left(\frac{y}{x}\right)$

∴ log u is a homogeneous function in x and y of degree 2.
Thus, by Euler's theorem, we have

$$x\frac{\partial}{\partial x}(\log u) + y\frac{\partial}{\partial y}(\log u) = 2\log u$$

$$\Rightarrow \quad x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2u\log u.$$

Q 35. If $u = x+y+z$, $uv = y+z$, $uvw = z$ show that $\frac{\partial(x,y,z)}{\partial(u,v,w)} = u^2v$. (PTU, Dec. 2009)

Solution. Given, $u = x+y+z$ ...(i)

$uv = y+z$ ...(ii)

$uvw = z$ ...(iii)

(i) − (ii) ⇒ $u - uv = x$

(ii) − (iii) ⇒ $uv - uvw = y$

also $uvw = z$

Now

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

$$= \begin{vmatrix} 1-v & -u & 0 \\ v-vw & u-uw & -uv \\ vw & uw & uv \end{vmatrix}$$

operate $R_2 \to R_2 + R_3$

$$= \begin{vmatrix} 1-v & -u & 0 \\ v & u & 0 \\ vw & uw & uv \end{vmatrix}$$

$$= uv(u - uv + uv)$$

$$= u^2v$$

Q 36. What is homogeneous function? State Euler's theorem on homogeneous functions. (PTU, Dec. 2010)

Solution. A function f(x, y, z) is said to be homogeneous in x, y, z of degree n

if $$f(x,y,z) = x^n\phi\left(\frac{y}{x},\frac{z}{x}\right)$$

e.g. $$f(x,y) = x^2+y^2 = x^2\left(1+\frac{y^2}{x^2}\right) = x^2\phi\left(\frac{y}{x}\right)$$

It is a homogeneous function in x, y of degree 2.

Euler's Theorem : If f be a homogeneous function in x, y and z of degree n

Then $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} = nf$

Q 37. Show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2u\log u$ where $\log u = \frac{x^3+y^3}{3x+4y}$. (PTU, Dec. 2010)

Solution. Given, $\log u = \frac{x^3+y^3}{3x+4y} = \frac{x^3\left[1+\left(\frac{y}{x}\right)^3\right]}{x\left[3+4\frac{y}{x}\right]} = x^2\phi\left(\frac{y}{x}\right)$

∴ log u is a homogeneous function in x and y of degree 2.
Thus by Euler's theorem,

$$x\frac{\partial}{\partial x}(\log u) + y\frac{\partial}{\partial y}(\log u) = 2\log u$$

i.e. $$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2u\log u.$$

Q 38. If $z = f(x,y)$ is a surface, then what is Geometrical meaning of $\frac{\partial z}{\partial x}$ (partial derivative w.r.t. x). (PTU, May 2011)

Solution. Let z = f(x, y) be a function of two variables x and y and it represents a surface. Now this surface meets a plane parallel to xz plane i.e. y = b in z = f(x, b).

Now z = f(x, b) is a function of one variable and represents a curve.

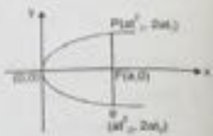
Also $\frac{d}{dx}$ [f(x, b)] at x = a represents the slope of the tangent to the curve at point (a, f(a, b))

$$f_x = \frac{d}{dx}[f(x,b)] \text{ at } x = a$$

$$\text{radius of curvature } \rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}} = \frac{\left[1+\frac{4a^2}{y^2}\right]^{3/2}}{\frac{-4a^2}{y^3}}$$

$$\rho = \frac{\left(y^2+4a^2\right)^{3/2}}{4a^2} = \frac{\left(4ax+4a^2\right)^{3/2}}{4a^2}$$

$$\rho \text{ (In magnitude)} = \frac{2}{\sqrt{a}}(x+a)^{3/2}$$



So $\rho = \frac{2}{\sqrt{a}}(x+a)^{\frac{3}{2}}$

Let PQ be the focal chord s.t. $t_1t_2 = -1$

$$\rho_1 = \frac{2}{\sqrt{a}}\left(at_1^2+a\right)^{\frac{3}{2}} = 2a\left(t_1^2+1\right)^{\frac{3}{2}}$$

and $\rho_2 = \frac{2}{\sqrt{a}}\left(at_2^2+a\right)^{\frac{3}{2}} = 2a\left(t_2^2+1\right)^{\frac{3}{2}}$

Now $(\rho_1)^{-\frac{2}{3}} + (\rho_2)^{-\frac{2}{3}} = (2a)^{-\frac{2}{3}}\left[\left(t_1^2+1\right)^{-1}+\left(t_2^2+1\right)^{-1}\right]$

$$= (2a)^{-\frac{2}{3}}\left[\frac{1}{t_1^2+1}+\frac{1}{1+t_2^2}\right]$$

$$= (2a)^{-\frac{2}{3}}\left[\frac{1}{1+t_1^2}+\frac{t_1^2}{1+t_1^2}\right] \qquad \left(\because t_2 = \frac{-1}{t_1}\right)$$

$$= (2a)^{-\frac{2}{3}}\left[\frac{1}{1+t_1^2}+\frac{t_1^2}{t_1^2+1}\right]$$

$$= (2a)^{-\frac{2}{3}}$$

Q 31. State and prove Euler's theorem. (PTU, May 2009)

Ans. Let z be a Homogeneous function of x and y of degree n

So $u = x^n f\left(\frac{y}{x}\right)$ ...(1)

$$\frac{\partial u}{\partial x} = x^n f'\left(\frac{y}{x}\right)\left(\frac{-y}{x^2}\right) + f\left(\frac{y}{x}\right) nx^{n-1}$$

$$\frac{\partial u}{\partial y} = x^n f'\left(\frac{y}{x}\right)\frac{1}{x}$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = x\left[-yx^{n-2}f'\left(\frac{y}{x}\right) + nx^{n-1}f\left(\frac{y}{x}\right)\right] + yx^{n-1}f'\left(\frac{y}{x}\right)$$

$$= nx^n f\left(\frac{y}{x}\right) = nu \qquad \text{(using (1))}$$

Q 32. If $z = \sqrt{x^2+y^2}$ and $x^3+y^3+3axy = 5a^2$, find the value of $\frac{dz}{dx}$, when $x = y = a$. (PTU, May 2009)

Ans. Given $z = \sqrt{x^2+y^2}$ ...(*) and $f(x, y) = x^3+y^3+3axy-5a^2 = 0$ ...(1)

Here z is a function of x and y, we have

$$\frac{dz}{dx} = \frac{\partial z}{\partial x}\frac{dx}{dx} + \frac{\partial z}{\partial y}\frac{dy}{dx} \qquad ...(2)$$

Differentialy partially eq (1) w.r.t. x and y, we get

$$f_x = 3x^2+3ay;\ f_y = 3y^2+3ax$$

$$\frac{dy}{dx} = -\frac{f_x}{f_y} = -\frac{x^2+ay}{y^2+ax} \qquad ...(3)$$

also diff. (*) w.r.t. x and y partially, we get

$$\frac{\partial z}{\partial x} = \frac{x}{\sqrt{x^2+y^2}};\ \frac{\partial z}{\partial y} = \frac{y}{\sqrt{x^2+y^2}} \qquad ...(4)$$

putting eqs. (3) and (4) in eq (2); we get

$$\frac{dz}{dx} = \frac{x}{\sqrt{x^2+y^2}} - \frac{y}{\sqrt{x^2+y^2}}\cdot\frac{x^2+ay}{y^2+ax}$$

at $x = y = a$, $\frac{dz}{dx} = \frac{a}{\sqrt{2a^2}} - \frac{a}{\sqrt{2a^2}}\cdot\frac{a^2+a^2}{a^2+a^2} = 0$

Q 33. If $x = r\cos\theta$, $y = r\sin\theta$, then show that $\frac{\partial(r,\theta)}{\partial(x,y)} = \frac{1}{r}$. (PTU, Dec. 2010; May 2013)

Solution. Now, $\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$; $\frac{\partial(r,\theta)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{vmatrix}$

Now, $x = r\cos\theta,\ y = r\sin\theta$

$$\sqrt{x^2+y^2} = r \ \&\ \tan\theta = \frac{y}{x}$$

i.e. $\theta = \tan^{-1}\frac{y}{x}$

$$V^2 = 64\, x^2 y^2 z^2 = 64\, x^2 y^2 c^2 \left[1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right]$$

$\Rightarrow$ $$f(x, y) = V^2 = 64c^2\left[x^2y^2 - \frac{x^4y^2}{a^2} - \frac{x^2y^4}{b^2}\right]$$

$\Rightarrow$ $$f_x = 64c^2\left[2xy^2 - \frac{4x^3y^2}{a^2} - \frac{2xy^4}{b^2}\right]$$

and $$f_y = 64c^2\left[2x^2y - \frac{2x^4y}{a^2} - \frac{4x^2y^3}{b^2}\right]$$

$$A = f_{xx} = 64c^2\left[2y^2 - \frac{12x^2y^2}{a^2} - \frac{2y^4}{b^2}\right]$$

$$C = f_{yy} = 64c^2\left[2x^2 - \frac{2x^4}{a^2} - \frac{12x^2y^2}{b^2}\right]$$

$$B = f_{xy} = 64c^2\left[4xy - \frac{8x^3y}{a^2} - \frac{8xy^3}{b^2}\right]$$

For max. or min. $f_x = 0$ and $f_y = 0$

$$128c^2xy^2\left[1 - \frac{2x^2}{a^2} - \frac{y^2}{b^2}\right] = 0 \text{ and } 128c^2x^2y\left[1 - \frac{x^2}{a^2} - \frac{2y^2}{b^2}\right] = 0 \quad \text{.....(2)}$$

Since $x \neq 0, y \neq 0$ eq (1) and (2) gives

$$1 - \frac{2x^2}{a^2} - \frac{y^2}{b^2} = 0 \text{ .....(3) and } 1 - \frac{x^2}{a^2} - \frac{2y^2}{b^2} = 0 \text{ .....(4)}$$

Subtracting (3) and (4) we get

$$-\frac{x^2}{a^2} + \frac{y^2}{b^2} = 0 \Rightarrow y = \pm\frac{b}{a}x$$

$$y = \frac{b}{a}x \qquad [\because y > 0]$$

$$\text{eq (3) gives, } 1 - \frac{2x^2}{a^2} - \frac{x^2}{a^2} = 0 \Rightarrow \frac{3x^2}{a^2} = 1 \Rightarrow x = \pm\frac{a}{\sqrt{3}} \qquad [\because x > 0]$$

$$x = \frac{a}{\sqrt{3}}$$

$$y = \frac{b}{\sqrt{3}} \quad (*) \text{ gives } \frac{z^2}{c^2} = 1 - \frac{a^2}{3}\cdot\frac{1}{a^2} - \frac{b^2}{3}\cdot\frac{1}{b^2}$$

$\Rightarrow$ $$z = \frac{c}{\sqrt{3}} \qquad [\because z > 0]$$

$\left(\frac{a}{\sqrt{3}}, \frac{b}{\sqrt{3}}\right)$ is a stationary point

Now $$A = 64c^2\left[\frac{2b^2}{3} - \frac{12}{a^2}\left(\frac{a^2}{3}\right)\left(\frac{b^2}{3}\right) - \frac{2}{b^2}\left(\frac{b^4}{9}\right)\right]$$

$$= \frac{-512}{9}b^2c^2 < 0$$

$$B = 64c^2\left[4\cdot\frac{ab}{3} - \frac{8}{a^2}\cdot\frac{a^3}{3\sqrt{3}}\cdot\frac{b}{\sqrt{3}} - \frac{8}{b^2}\cdot\frac{a}{\sqrt{3}}\cdot\frac{b^3}{3\sqrt{3}}\right] = \frac{-256}{9}abc^2$$

$$C = \frac{-512}{9}a^2c^2$$

$$AC - B^2 = \left(\frac{256}{9}\right)^2 a^2b^2c^4 \times 3 > 0 \text{ also } A < 0$$

So V^2 is maximum Hence V is maximum

$$\text{Maximum volume} = V = 8\cdot\frac{a}{\sqrt{3}}\cdot\frac{b}{\sqrt{3}}\cdot\frac{c}{\sqrt{3}} = \frac{8abc}{3\sqrt{3}}$$

Q 33. The sum of three positive numbers is constant. Prove that their product is maximum when they are equal. (PTU, Dec. 2006)

Solution. Let the three numbers are x, y, z where x, y, z > 0

$$\text{s.t. } x + y + z = K \qquad \text{.....(1)}$$

Let $P = xyz = xy(K - x - y)$ (using eq (1))

$$\frac{\partial P}{\partial x} = y(K - 2x - y)$$

$$\frac{\partial P}{\partial y} = x(K - x - 2y)$$

For maxima or minima $\frac{\partial P}{\partial x} = 0 = \frac{\partial P}{\partial y}$

$$y(K - 2x - y) = 0 \text{ .....(2) and } x(K - x - 2y) = 0 \text{ .....(3)}$$

on solving eq (2) and (3) we get

$$x + 2y = 2x + y = K \Rightarrow x = y = \frac{K}{3} \quad (\because x > 0 \text{ and } y > 0)$$

eq (1) gives

$$Z = K - x - y = K - \frac{2K}{3} = \frac{K}{3}$$

$$x = y = z = \frac{K}{3}$$

∴ eq. of normal to the given surface (1) at (2, 2, 6) is

$$\frac{x-2}{\frac{\partial f}{\partial x}} = \frac{y-2}{\frac{\partial f}{\partial y}} = \frac{z-6}{\frac{\partial f}{\partial z}}$$

i.e. $$\frac{x-2}{16} = \frac{y-2}{16} = \frac{z-6}{-12}$$

i.e. $$\frac{x-2}{4} = \frac{y-2}{4} = \frac{z-6}{-3}$$

Q.50. A rectangular box open at the top is to have the volume of 32 cubic feet. Find the dimensions of the box requiring least material for its construction. **(PTU, May 2008)**

Solution. Let x, y, z be the dimensions of the box.

$x, y, z > 0$

again given $32 = xyz$

S = Surface area $= xy + 2yz + 2zx$

Now $$S = xy + 2(x+y)\frac{32}{xy} = xy + 64\left[\frac{1}{y} + \frac{1}{x}\right]$$

for extreme values, $$\frac{\partial S}{\partial x} = 0,\ \frac{\partial S}{\partial y} = 0$$

$$\frac{\partial S}{\partial x} = 0 \Rightarrow y + 64\left(\frac{-1}{x^2}\right) = 0 \quad \text{—(1)}$$

$$\frac{\partial S}{\partial y} = 0 \Rightarrow x + 64\left(\frac{-1}{y^2}\right) = 0 \quad \text{—(2)}$$

from (1), $x^2y = 64$ —(3), from (2) $xy^2 = 64$ —(4)

Dividing (3) and (4), we get

$$\frac{x}{y} = 1 \Rightarrow x = y \quad \text{from (3) we have } x^3 = 64 \Rightarrow x = 4 = y$$

again $xyz = 32 \Rightarrow z = 2$

Now $$f_{xx} = \frac{\partial^2 S}{\partial x^2} = \frac{128}{x^3},\ \frac{\partial^2 S}{\partial y^2} = \frac{128}{y^3},\ \frac{\partial^2 S}{\partial x \partial y} = 1$$

$$A = \frac{128}{64} = 2,\ B = 1,\ C = 2$$

Now $AC - B^2 = 4 - 1 = 3 > 0,\ A = 2 > 0$

∴ S is Minimum for $x = 4 = y,\ z = 2$

Q.51. Find the maxima and minima of $f(x, y) = x^3y^2(1-x-y)$. **(PTU, May 2009; Dec. 2007; June 2007)**

Solution. $$f(x, y) = x^3y^2(1-x-y) = x^3y^2 - x^4y^2 - x^3y^3$$

$$f_x = 3x^2y^2 - 4x^3y^2 - 3x^2y^3;\ f_y = 2x^3y - 2x^4y - 3x^3y^2$$
$$f_{xx} = 6xy^2 - 12x^2y^2 - 6xy^3,\ f_{yy} = 2x^3 - 2x^4 - 6x^3y$$
$$f_{xy} = 6x^2y - 8x^3y - 9x^2y^2$$

For maxima or minima put $f_x = 0 = f_y$

⇒ $x^2y^2(3 - 4x - 3y) = 0$ —(1)

and $x^3y(2 - 2x - 3y) = 0$ —(2)

from (1) and (2) the possible solution is given by

$$\left.\begin{array}{l} 4x + 3y = 3 \\ 2x + 3y = 2 \end{array}\right] \Rightarrow x = \frac{1}{2},\ y = \frac{1}{3} \quad \left(\frac{1}{2}, \frac{1}{3}\right) \text{ is a stationary point}$$

$$A = (f_{xx})_{\left(\frac{1}{2},\frac{1}{3}\right)} = 6\cdot\frac{1}{2}\cdot\frac{1}{9} - 12\cdot\frac{1}{4}\cdot\frac{1}{9} - 6\cdot\frac{1}{2}\cdot\frac{1}{27}$$
$$= \frac{1}{3} - \frac{1}{3} - \frac{1}{9} = -\frac{1}{9} < 0$$

$$B = (f_{xy})_{\left(\frac{1}{2},\frac{1}{3}\right)} = 6\cdot\frac{1}{4}\cdot\frac{1}{3} - 8\cdot\frac{1}{8}\cdot\frac{1}{3} - 9\cdot\frac{1}{4}\cdot\frac{1}{9} = \frac{1}{2} - \frac{1}{3} - \frac{1}{4} = \frac{-1}{12}$$

$$C = (f_{yy})_{\left(\frac{1}{2},\frac{1}{3}\right)} = 2\cdot\frac{1}{8} - 2\cdot\frac{1}{16} - 6\cdot\frac{1}{8}\cdot\frac{1}{3} = \frac{1}{4} - \frac{1}{8} - \frac{1}{4} = \frac{-1}{8}$$

$$AC - B^2 = \frac{1}{72} - \frac{1}{144} = \frac{1}{144} > 0$$

∴ $\left(\frac{1}{2}, \frac{1}{3}\right)$ is a point of maxima and max. value $= \frac{1}{8}\cdot\frac{1}{9}\left(1 - \frac{1}{2} - \frac{1}{3}\right)$

i.e. max. value $= \frac{1}{72}\left(\frac{1}{6}\right) = \frac{1}{432}$

Now, $f(h, k) - f(0, 0) = h^3k^2(1 - h - k)$

For small values of h and k in the neighbourhood of (0, 0)

We have $f(h, k) - f(0, 0) = h^3k^2 > 0$ if $h > 0$ and < 0 if $h < 0$

∴ (0, 0) is a point of neither maxima nor minima.

Q.52. Find the volume of greatest rectangular parallelopiped that can be inscribed in the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$. **(PTU, May 2007)**

Solution. Let (x, y, z) be the vertex of the parallelopiped that is lies in the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \quad \text{—(*)}$$

Let 2x, 2y, 2z be the dimensions of the || piped

∴ Volume of the || piped = $V = (2x)(2y)(2z) = 8xyz$

We have to maximum V so it is convenient to maximise V^2.

and $\quad z - x - 2y = 0$

$f(x, y) = x^2 + y^2 + 10 - x - 2y$

For maxima and minima $\frac{\partial f}{\partial x} = 0 = \frac{\partial f}{\partial y}$

$$\Rightarrow \quad \frac{\partial f}{\partial x} = 0 \Rightarrow 2x - 1 = 0 \Rightarrow x = \frac{1}{2}$$

$$\text{and} \quad \frac{\partial f}{\partial y} = 0 \Rightarrow 2y - 2 = 0 \Rightarrow y = 1$$

$$z = x + 2y = \frac{1}{2} + 2 = \frac{5}{2}$$

point of maxima and minima is given by $\left(\frac{1}{2}, 1, \frac{5}{2}\right)$

Now $\quad A = \left(f_{xx}\right)_{\left(\frac{1}{2}, 1\right)} = 2;\ B = \left(f_{xy}\right)_{\left(\frac{1}{2}, 1\right)} = 0$

and $\quad C = \left(f_{yy}\right)_{\left(\frac{1}{2}, 1\right)} = 2$

$$AC - B^2 = 4 > 0 \text{ and } A = 2 > 0$$

$\left(\frac{1}{2}, 1, \frac{5}{2}\right)$ is a point of minima.

Q 45. What do you understand by a level surface? Illustrate with the help of one example. **(PTU, May 2005)**

Solution. Level surface : Let the equation of surface be $\phi(x, y, z) = c$

If this surface be drawn through any point P s.t. at each point on it, the function has the same value as at point P. Then such a surface is called level surface of function $\phi(x, y, z)$ through P.

The equipotential or Isothermal surface is the level surface.

Q 46. Find the equation of the tangent plane of the surface $x^3 + y^3 + 3xyz = 3$ at (1, 2, −1). **(PTU, May 2006)**

Solution. $\quad f(x, y, z) = x^3 + y^3 + 3xyz - 3 = 0 \quad$ —(1)

$f_x = 3x^2 + 3yz;\ f_y = 3y^2 + 3xz;\ f_z = 3yx$

$(f_x)_{(1,2,-1)} = 3 - 6 = -3$ and $(f_z)_{(1,2,-1)} = 6$

$(f_y)_{(1,2,-1)} = 12 - 3 = 9$

eq. of tangent plane to surface (1) at (1, 2, −1)

is given by $(x - 1)\frac{\partial f}{\partial x} + (y - 2)\frac{\partial f}{\partial y} + (z + 1)\frac{\partial f}{\partial z} = 0$

$\Rightarrow -3(x - 1) + 9(y - 2) + 6(z + 1) = 0$

$\Rightarrow \quad x - 3y - 2z = -3$

Q 47. Find the equation of the normal line to the surface $xyz = a^3$ at (x_1, y_1, z_1). **(PTU, Dec. 2005)**

Solution. The given surface be $F(x, y, z) = xyz - a^3 = 0$ —(1)

$$\frac{\partial F}{\partial x} = yz \Rightarrow \left(\frac{\partial F}{\partial x}\right)_{(x_1, y_1, z_1)} = y_1 z_1$$

$$\frac{\partial F}{\partial y} = xz \Rightarrow \left(\frac{\partial F}{\partial y}\right)_{(x_1, y_1, z_1)} = x_1 z_1$$

$$\frac{\partial F}{\partial z} = xy \Rightarrow \left(\frac{\partial F}{\partial z}\right)_{(x_1, y_1, z_1)} = x_1 y_1$$

eq. of normal line through (x_1, y_1, z_1) to the surface is given by

$$\frac{x - x_1}{y_1 z_1} = \frac{y - y_1}{x_1 z_1} = \frac{z - z_1}{x_1 y_1}$$

Q 48. Find the equation of the tangent plane to the surface $xyz = a^3$ at (x_1, y_1, z_1). **(PTU, May 2005)**

Solution. The given surface be $f(x, y, z) = xyz - a^3 = 0$

$$\frac{\partial f}{\partial x} = yz \Rightarrow \left(\frac{\partial f}{\partial x}\right)_{(x_1, y_1, z_1)} = y_1 z_1$$

$$\frac{\partial f}{\partial y} = xz \Rightarrow \left(\frac{\partial f}{\partial y}\right)_{(x_1, y_1, z_1)} = x_1 z_1$$

$$\frac{\partial f}{\partial z} = xy \Rightarrow \left(\frac{\partial f}{\partial z}\right)_{(x_1, y_1, z_1)} = x_1 y_1$$

∴ eq. of tangent plane to the surface $xyz - a^3 = 0$ is given by

$(x - x_1)\frac{\partial f}{\partial x} + (y - y_1)\frac{\partial f}{\partial y} + (z - z_1)\frac{\partial f}{\partial z} = 0$

$\Rightarrow \quad (x - x_1)(y_1 z_1) + (y - y_1)(x_1 z_1) + (z - z_1)(x_1 y_1) = 0$

$\Rightarrow \quad xy_1 z_1 + yx_1 z_1 + zx_1 y_1 = 3x_1 y_1 z_1$

$\Rightarrow \quad \frac{x}{x_1} + \frac{y}{y_1} + \frac{z}{z_1} = 3$ is the req. eq. of tangent plane.

Q 49. Find the equations of the normal to the surface $z^2 = 4(1 + x^2 + y^2)$ at (2, 2, 6). **(PTU, Dec. 2006)**

Solution. Given eq. of surface be

$f(x, y, z) = 4(1 + x^2 + y^2) - z^2 = 0 \quad$ —(1)

$$\frac{\partial f}{\partial x} = 8x;\ \frac{\partial f}{\partial y} = 8y;\ \frac{\partial f}{\partial z} = -2z$$

Thus $\left(\frac{\partial f}{\partial x}\right)_{(2,2,6)} = 16;\ \left(\frac{\partial f}{\partial y}\right)_{(2,2,6)} = 16;\ \left(\frac{\partial f}{\partial z}\right)_{(2,2,6)} = -12$

Hence the shortest distance between $= \sqrt{36+9} = 3\sqrt{5}$
line and ellipse is $\sqrt{5}$

Q 58. If $x+y+z=1$, prove that stationary value of $u = \frac{a^3}{x^2}+\frac{b^3}{y^2}+\frac{c^3}{z^2}$ is given by:

$$x = \frac{a}{a+b+c},\ y = \frac{b}{a+b+c},\ z = \frac{c}{a+b+c}$$ **(PTU, May 2004)**

Solution. $$u = \frac{a^3}{x^2}+\frac{b^3}{y^2}+\frac{c^3}{z^2};\ x+y+z=1 \quad \text{—— (1)}$$

Therefore Lagrange's function is given by

$$F(x,y,z) = \frac{a^3}{x^2}+\frac{b^3}{y^2}+\frac{c^3}{z^2}+\lambda(x+y+z-1)$$

Where λ = Lagrange's multiplier

For maxima or minima we put $\frac{\partial F}{\partial x} = 0 = \frac{\partial F}{\partial y} = \frac{\partial F}{\partial z}$

$$\frac{\partial F}{\partial x} = \frac{-2a^3}{x^3}+\lambda = 0 \quad \text{—— (2)}$$

$$\frac{\partial F}{\partial y} = \frac{-2b^3}{y^3}+\lambda = 0 \quad \text{—— (3)}$$

$$\frac{\partial F}{\partial z} = \frac{-2c^3}{z^3}+\lambda = 0 \quad \text{—— (4)}$$

from (2), (3) and (4) we get

$$\lambda = \frac{2a^3}{x^3} = \frac{2b^3}{y^3} = \frac{2c^3}{z^3}$$

$\Rightarrow$ $$\frac{\lambda}{2} = \frac{a^3}{x^3} = \frac{b^3}{y^3} = \frac{c^3}{z^3}$$

$\Rightarrow$ $$\frac{a}{x} = \frac{b}{y} = \frac{c}{z} = \left(\frac{\lambda}{2}\right)^{1/3} = K$$

$\Rightarrow$ $$x = \frac{a}{K};\ y = \frac{b}{K};\ z = \frac{c}{K}$$

eq. (1) gives

$$\frac{a}{K}+\frac{b}{K}+\frac{c}{K} = 1 \Rightarrow K = a+b+c$$

$$x = \frac{a}{a+b+c};\ y = \frac{b}{a+b+c};\ z = \frac{c}{a+b+c}$$

Q 59. If $u = ax^2+by^2+cz^2$ where $x^2+y^2+z^2=1$, and $lx+my+nz=0$, prove that the stationary values of u satisfy the equation : $\frac{l^2}{a-u}+\frac{m^2}{b-u}+\frac{n^2}{c-u}=0$. **(PTU, Dec. 2005)**

Solution. Let $u = ax^2+by^2+cz^2$ subject to constraints
$x^2+y^2+z^2=1$ —— (1) and $lx+my+nz=0$
Lagrange's function $F(x,y,z) = ax^2+by^2+cz^2+\lambda_1(x^2+y^2+z^2-1)+\lambda_2(lx+my+nz)$ —— (2)

For max. or minima $\frac{\partial F}{\partial x} = 0 = \frac{\partial F}{\partial y} = \frac{\partial F}{\partial z}$

$$\frac{\partial F}{\partial x} = 2ax+\lambda_1(2x)+\lambda_2 l = 0 \quad \text{—— (3)}$$

$$\frac{\partial F}{\partial y} = 2by+\lambda_1(2y)+\lambda_2 m = 0 \quad \text{—— (4)}$$

$$\frac{\partial F}{\partial z} = 2cz+\lambda_1(2z)+\lambda_2 n = 0 \quad \text{—— (5)}$$

Multiply eq (3) by x, eq (4) by y and eq (5) by z
and adding we get
$2(ax^2+by^2+cz^2)+2\lambda_1(x^2+y^2+z^2)+\lambda_2(lx+my+nz)=0$
$\Rightarrow$ $$2u+2\lambda_1 = 0 \Rightarrow \lambda_1 = -u$$

$\therefore$ eq (3) gives ; $2ax-2ux+\lambda_2 l = 0 \Rightarrow x = \frac{\lambda_2 l}{2(u-a)}$

eq (4) and (5) gives ; $y = \frac{\lambda_2 m}{2(u-b)}$ and $z = \frac{\lambda_2 n}{2(u-c)}$

$\therefore$ eq (2) gives

$$\frac{-\lambda_2}{2}\left[\frac{l^2}{a-u}+\frac{m^2}{b-u}+\frac{n^2}{c-u}\right] = 0$$

[Now $\lambda_2 \neq 0$ $\because$ if $\lambda_2 = 0 \Rightarrow x = y = z = 0$]

$$\frac{l^2}{a-u}+\frac{m^2}{b-u}+\frac{n^2}{c-u} = 0 \text{ gives the stationary values of u.}$$

∴ eq (1) gives

$$4x^3 - 4x - 4x = 0 \Rightarrow 4x^3 - 8x = 0 \Rightarrow 4x(x^2 - 2) = 0$$

$$\Rightarrow \quad x = 0, \sqrt{2}, -\sqrt{2} \quad \therefore y = 0, -\sqrt{2}, \sqrt{2}$$

∴ Stationary points are $(0, 0); (\sqrt{2}, -\sqrt{2}); (-\sqrt{2}, \sqrt{2})$

$$\text{Now } \frac{\partial^2 f}{\partial x^2} = 12x^2 - 4; \frac{\partial^2 f}{\partial y^2} = 12y^2 - 4; \frac{\partial^2 f}{\partial x \partial y} = 4$$

Case-I. at (0, 0)

$$A = \left(\frac{\partial^2 f}{\partial x^2}\right)_{(0,0)} = -4, B = \left(\frac{\partial^2 f}{\partial x \partial y}\right)_{(0,0)} = 4; C = \left(\frac{\partial^2 f}{\partial y^2}\right)_{(0,0)} = -4$$

$$AC - B^2 = 16 - 16 = 0$$

∴ (0, 0) is a point of further discussion

Now $f(x, y) = x^4 + y^4 - 2(x - y)^2$

$$f(h, k) = h^4 + k^4 - 2(h - k)^2$$

when $h = k$

$$f(h, k) = 2h^4 > 0 = f(0, 0)$$

also when $h \neq k$

$$f(h, k) = -2(h - k)^2 < 0 = f(0, 0)$$

[Where h, k are so small s.t. h^4 and k^4 are neglected]

So In the neighbourhood of (0, 0) there are some points where $f(h, k) < f(0, 0)$ and some points $f(h, k) > f(0, 0)$ ∴ (0, 0) is not an extreme point.

Case-II. at $(\sqrt{2}, -\sqrt{2})$

$$A = \left(\frac{\partial^2 f}{\partial x^2}\right)_{(\sqrt{2},-\sqrt{2})} = 20; B = 4; C = \left(\frac{\partial^2 f}{\partial y^2}\right)_{(\sqrt{2},-\sqrt{2})} = 20$$

$$AC - B^2 = 400 - 16 = 384 > 0$$

and $A > 0$

∴ $(\sqrt{2}, -\sqrt{2})$ is a point of minima and min. value $= 4 + 4 - 4 - 8 - 4 = -8$

Case-III. at $(-\sqrt{2}, \sqrt{2})$

$$A = 20; B = 4, C = 20$$

$$AC - B^2 = 384 > 0; A = 20 > 0$$

∴ $(-\sqrt{2}, \sqrt{2})$ is a point of minima and minimum value $= 4 + 4 - 4 - 8 - 4 = -8$.

Q 57. Find the shortest distance between the line $y = 10 - 2x$ and the ellipse

$$\frac{x^2}{4} + \frac{y^2}{9} = 1.$$

(PTU, Dec. 2004)

Solution. Let (x, y) be any point on given ellipse and (u, v) be any point on given line we have to minimise or maximise

$d = \sqrt{(x - u)^2 + (y - v)^2}$ as it is convenient to minimise or maximise

$d^2 = f(x, y, u, v) = (x - u)^2 + (y - v)^2$

Subject to constraints $\phi_1(x, y) = \frac{x^2}{4} + \frac{y^2}{9} - 1 = 0$ ...(1); $\phi_2(u, v) = 2u + v - 10 = 0$ ...(2)

Let us form $F(x, y, u, v) = (x - u)^2 + (y - v)^2 + \lambda_1\left(\frac{x^2}{4} + \frac{y^2}{9} - 1\right) + \lambda_2(2u + v - 10)$

where λ_1, λ_2 are lagrange's multipliers

For extreme values we put $\frac{\partial F}{\partial x} = \frac{\partial F}{\partial y} = \frac{\partial F}{\partial u} = \frac{\partial F}{\partial v} = 0$

i.e.

$$\frac{\partial F}{\partial x} = 2(x - u) + \frac{2\lambda_1 x}{4} = 0 \Rightarrow \lambda_1 x = 4(u - x) \quad ...(3)$$

$$\frac{\partial F}{\partial y} = 2(y - v) + \frac{2\lambda_1 y}{9} = 0 \Rightarrow \lambda_1 y = 9(v - y) \quad ...(4)$$

$$\frac{\partial F}{\partial u} = -2(x - u) + 2\lambda_2 = 0 \Rightarrow \lambda_2 = (x - u) \quad ...(5)$$

$$\frac{\partial F}{\partial v} = -2(y - v) + \lambda_2 = 0 \Rightarrow \lambda_2 = 2(y - v) \quad ...(6)$$

From (5) and (6); we have; $(x - u) = 2(y - v)$ ...(7)

From (3) and (4); we have; $4(u - x)y = 9x(v - y)$ ...(8)

Dividing (7) and (8); we get

$$\frac{-1}{4y} = \frac{-2}{9x} \Rightarrow 9x = 8y \quad ...(9)$$

∴ From (1); $x = \pm\frac{8}{5}$ and $y = \pm\frac{9}{5}$

When $x = \frac{8}{5}, y = \frac{9}{5}$ from (7); $\frac{8}{5} - u = 2\left(\frac{9}{5} - v\right) \Rightarrow u = 2v - 2$

∴ From (2); we have $u = \frac{18}{5}$ and $v = \frac{14}{5}$

∴ required distance $= \sqrt{\left(\frac{8}{5} - \frac{18}{5}\right)^2 + \left(\frac{9}{5} - \frac{14}{5}\right)^2} = \sqrt{5}$

When $x = -\frac{8}{5}; y = -\frac{9}{5}$ ∴ from (7); $u - 2v = 2$ ∴ from (2); we have

$u = \frac{22}{5}, v = \frac{6}{5}$ ∴ req. distance $= \sqrt{\left(\frac{22}{5} + \frac{8}{5}\right)^2 + \left(\frac{6}{5} + \frac{9}{5}\right)^2}$

$\therefore \left(\frac{K}{3}, \frac{K}{3}\right)$ is a stationary point

$$\text{Now } \frac{\partial^2 P}{\partial x^2} = -2y \; ; \; \frac{\partial^2 P}{\partial x \partial y} = K - 2x - 2y \; ; \; \frac{\partial^2 P}{\partial y^2} = -2x$$

$$A = \left(\frac{\partial^2 P}{\partial x^2}\right)_{\left(\frac{K}{3}, \frac{K}{3}\right)} = \frac{-2K}{3} \; ; \; B = \left(\frac{\partial^2 P}{\partial x \partial y}\right)_{\left(\frac{K}{3}, \frac{K}{3}\right)} = \frac{-K}{3}$$

$$C = \left(\frac{\partial^2 P}{\partial y^2}\right)_{\left(\frac{K}{3}, \frac{K}{3}\right)} = -\frac{2K}{3}$$

$$\text{Now } AC - B^2 = \frac{4K^2}{9} - \frac{K^2}{9} = \frac{3K^2}{9} > 0 \text{ also } A = -\frac{2K}{3} < 0 \; [\because K > 0]$$

$\left(\frac{K}{3}, \frac{K}{3}\right)$ is a point of maxima and P is maximum when $x = y = z = \frac{K}{3}$.

Q 54. Find the maximum and minimum values of $x^3 + y^3 - 3axy$. (PTU, May 2006)

Solution. Let $f(x, y) = x^3 + y^3 - 3axy$

$\Rightarrow \quad f_x = 3x^2 - 3ay \; ; \; f_y = 3y^2 - 3ax \; ; \; f_{xx} = 6x \; ; \; f_{yy} = 6y$

and $\quad f_{xy} = -3a = f_{yx}$

For maxima or minima we put $f_x = 0 \; ; \; f_y = 0$

$\Rightarrow \quad x^2 - ay = 0$ ..... (1) and $y^2 - ax = 0$ ..... (2)

from (1) and (2), we have

$$y = \frac{x^2}{a} \Rightarrow \text{eq (2) gives } \frac{x^4}{a^2} - ax = 0 \Rightarrow x(x^3 - a^3) = 0 \Rightarrow x = 0, a$$

$\Rightarrow \quad y = 0, a \quad \therefore$ Stationary points are given by (0, 0) and (a, a)

Case-I at (0, 0)

$$A = (f_{xx})_{(0,0)} = 0, \; B = f_{xy} \text{ at } (0, 0) = -3a, \; C = f_{yy} \text{ at } (0,0) = 0$$

$$AC - B^2 = 0 - 9a^2 = -9a^2 < 0$$

(0, 0) is a point of neither maxima and minima.

Case-II at (a, a)

$$A = 6a, \; B = -3a, \; C = 6a$$

$$AC - B^2 = 36a^2 - 9a^2 = 27a^2 > 0$$

Now (a, a) is a point of maxima if $a < 0$ and is a point of minima if $a > 0$

i.e. maximum or minimum values $= a^3 + a^3 - 3a^3 = -a^3$

Q 55. Use Lagrange's method to find the minimum value of $x^2 + y^2 + z^2$ subject to the conditions $x + y + z = 1$ and $xyz + 1 = 0$. (PTU, Dec. 2005)

Solution. $f(x, y, z) = x^2 + y^2 + z^2$ subjects to constraints

$\phi(x, y, z) = x + y + z - 1 = 0$ ..... (1) and $\psi(x, y, z) = xyz + 1 = 0$ ..... (2)

Lagrange's function is given by

$$F(x, y, z) = f(x, y, z) + \lambda\phi(x, y, z) + \mu\psi(x, y, z)$$

where λ, μ are Lagranges multipliers.

$$F(x, y, z) = x^2 + y^2 + z^2 + \lambda(x + y + z - 1) + \mu(xyz + 1)$$

For max. or minima $\frac{\partial F}{\partial x} = 0 = \frac{\partial F}{\partial y} = \frac{\partial F}{\partial z}$

$$\frac{\partial F}{\partial x} = 2x + \lambda + \mu yz = 0 \quad \text{..... (3)}$$

$$\frac{\partial F}{\partial y} = 2y + \lambda + \mu xz = 0 \quad \text{..... (4)}$$

$$\frac{\partial F}{\partial z} = 2z + \lambda + \mu xy = 0 \quad \text{..... (5)}$$

Subtracting (4) and (5), we get

$2(y - z) + \mu x(z - y) = 0 \Rightarrow (y - z)(2 - \mu x) = 0$

$\Rightarrow \quad y = z \text{ or } \mu = \frac{2}{x}$ ..... (6)

again (3) – (4) gives

$2(x - y) + \mu(y - x)z = 0 \Rightarrow x - y = 0 \text{ or } \mu = \frac{2}{z}$ ..... (7)

from (6) and (7) gives

$x = y = z \text{ or } \mu = \frac{2}{x} = \frac{2}{y} = \frac{2}{z}$

$\therefore$ eq (1) gives : $3x - 1 = 0 \Rightarrow x = \frac{1}{3}$

$\therefore$ stationary point becomes $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$

$$\text{Now } d^2F = F_{xx}(dx)^2 + F_{yy}(dy)^2 + F_{zz}(dz)^2 + 2F_{xy}\,dx\,dy + 2F_{yz}\,dy\,dz + 2F_{zx}\,dz\,dx$$

$$= 2(dx)^2 + 2(dy)^2 + 2(dz)^2 + 2\mu z\,dx\,dy + 2\mu x\,dy\,dz + 2\mu y\,dz\,dx$$

$$= 2(dx)^2 + 2(dy)^2 + 2(dz)^2 + 4dx\,dy + 4\,dy\,dz + 4dz\,dx$$

$$= 2(dx + dy + dz)^2 > 0$$

$\therefore \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ is a point of minima and minimum value $= \frac{1}{9} + \frac{1}{9} + \frac{1}{9} = \frac{1}{3}$

Q 56. Locate the stationary points of : $x^4 + y^4 - 2x^2 + 4xy - 2y^2$ and decide about their nature. (PTU, May 2005)

Solution. Let $f(x, y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2$

$$\Rightarrow \quad \frac{\partial f}{\partial x} = 4x^3 - 4x + 4y \; ; \; \frac{\partial f}{\partial y} = 4y^3 + 4x - 4y$$

For maxima or minima $\frac{\partial f}{\partial x} = 0 = \frac{\partial f}{\partial y}$

$\Rightarrow \quad 4x^3 - 4x + 4y = 0$ ..... (1) and $4y^3 + 4x - 4y = 0$ ..... (2)

on adding (1) and (2), we get

$$4(x^3 + y^3) = 0 \Rightarrow x = -y$$

$$\frac{x}{y} = 2 \Rightarrow x = 2y$$

From eqn. (1), $y = 2$, therefore from Eqn. (1), $x = 4$ and $z = \frac{8}{xy} = \frac{8}{8} = 1$

Therefore, Eqn. (4, 2, 1) is the only stationary point.

$$f_{xx} = 40y\left[\frac{\left(x^2y+2xy^2+32\right)^2(-2xy)-2\left(32-x^2y\right)\left(x^2y+2xy^2+32\right)\left(2xy+2y^2\right)}{\left(x^2y+2xy^2+32\right)^4}\right]$$

$$= \frac{-80y}{\left(x^2y+2xy^2+32\right)^3}\left[x^3y^2+2x^2y^3+32xy+64xy+64y^2-2x^3y^2-2x^2y^3\right]$$

$$= \frac{-80y}{\left(x^2y+2xy^2+32\right)^3}\left[96xy+64y^2-x^3y^2\right]$$

$$f_{yy} = \frac{40x}{\left(x^2y+2xy^2+32\right)^4}\left[(x^2y+2xy^2+32)^2(-4xy)-2(32-2xy)^2+(x^2y+2xy^2+32)(x^2+4xy)\right]$$

$$= \frac{40x(-4)}{\left(x^2y+2xy^2+32\right)^3}\left[x^3y^2+2x^2y^3+32xy+16x^2+64xy-x^3y^2-4x^2y^3\right]$$

$$= \frac{-160x}{\left(x^2y+2xy^2+32\right)^3}\left[-2x^2y^3+96xy+16x^2\right]$$

$$f_{xy} = 40\left[\frac{\left(x^2y+2xy^2+32\right)^2\left(32-2x^2y\right)-2\left(32y-x^2y^2\right)\left(x^2y+2xy^2+32\right)\left(x^2+4xy\right)}{\left(x^2y+2xy^2+32\right)^4}\right]$$

$$= \frac{40}{\left(x^2y+2xy^2+32\right)^3}\left[(x^2y+2xy^2+32)(32-2x^2y)-2(32y-x^2y^2)(x^2+4xy)\right]$$

At (4, 2, 1), we have

$$A = f_{xx} = \frac{-80\times 4}{(32+32+32)^3}\left[96\times 4+64\times 2-64\times 2\right] = \frac{-80\times 16}{96\times 96} = \frac{-5}{36}$$

$$B = f_{xy} = \frac{40}{(96)^3}\left[96\times(-32)-2\times 0\right] = \frac{-40\times 32}{96\times 96} = \frac{-5}{36}$$

$$C = f_{yy} = \frac{-160\times 16\times 2}{(96)^3}\left[-32+96+32\right] = \frac{-160\times 16\times 2\times 96}{(96)^3} = \frac{-160\times 2}{6\times 96}$$

$$= \frac{-10\times 2}{36} = \frac{-20}{36}$$

Here, $AC - B^2 > 0$, $A < 0$

Therefore, (4, 2, 1) is a point of maxima thus $f(x, y, z)$ is maximum at $x = 4$, $y = 2$ and $z = 1$.

Q 66. Find the minimum value of the function $x^2 + y^2 + z^2$ subject to the condition $ax + by + cz = a + b + c$. (PTU, May 2011)

Solution. The given constraint is $ax + by + cz = a + b + c$ ...(1)

Therefore, $F(x, y, z) = x^2 + y^2 + z^2 + \lambda(ax + by + cz - a - b - c)$ ...(2)

For extreme value, we have

$$F_x = 2x + \lambda(a) = 0 \Rightarrow x = \frac{-\lambda a}{2}$$

$$F_y = 2y + \lambda(b) = 0 \Rightarrow y = \frac{-\lambda b}{2}$$

$$F_z = 2z + \lambda(c) = 0 \Rightarrow z = \frac{-\lambda c}{2}$$

Therefore, Eqn. (1) gives ; $\frac{-\lambda a^2}{2} \frac{-\lambda b^2}{2} \frac{-\lambda c^2}{2} = a + b + c \Rightarrow \lambda = \frac{-2(a+b+c)}{a^2+b^2+c^2}$

$$x = \frac{a(a+b+c)}{a^2+b^2+c^2},\; y = \frac{b(a+b+c)}{a^2+b^2+c^2},\; z = \frac{c(a+b+c)}{a^2+b^2+c^2}$$

Thus value of $x^2 + y^2 + z^2$

$$= \frac{a^2(a+b+c)^2}{\left(a^2+b^2+c^2\right)^2} + \frac{b^2(a+b+c)^2}{\left(a^2+b^2+c^2\right)^2} + \frac{c^2(a+b+c)^2}{\left(a^2+b^2+c^2\right)^2}$$

$$= \frac{(a+b+c)^2}{a^2+b^2+c^2}$$

Now we want to prove that this value is maximum or minimum.

For this, we have

$$d^2F = \Sigma F_{xx}(dx)^2 + 2\Sigma F_{xy}\,dx\,dy$$

Here

$$F_{xx} = 2 = F_{yy} = F_{zz},\; F_{xy} = 0 = F_{yz} = F_{zx}$$

$$d^2F = 2(dx)^2 + 2(dy)^2 + 2(dz)^2 > 0$$

Therefore, this value is minimum.

Q 67. Evaluate $\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dy\,dx}{1+x^2+y^2}$. (PTU, May 2006)

Ans. $$\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dy\,dx}{1+x^2+y^2} = \int_0^1 \frac{1}{\sqrt{1+x^2}}\left[\tan^{-1}\frac{y}{\sqrt{1+x^2}}\right]_0^{\sqrt{1+x^2}} dx$$

$$= \int_0^1 \frac{1}{\sqrt{1+x^2}} \tan^{-1} 1\, dx = \frac{\pi}{4}\int_0^1 \frac{dx}{\sqrt{1+x^2}}$$

$$= \frac{\pi}{4}\log\left[x+\sqrt{1+x^2}\right]_0^1 = \frac{\pi}{4}\log\left(1+\sqrt{2}\right)$$

put all these values in $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=1$, we have

$$\frac{1}{K}(a+b+c)=1 \Rightarrow K=a+b+c$$

$$x=\frac{2a}{a}, y=\frac{2a}{b}, z=\frac{2a}{c}$$

gives the stationary values of function.

Q 62. Find the equation of normal to the surface : $x^2+y^2+z^2=a^2$. **(PTU, Dec. 2009)**

Solution. Given surface be $F(x, y, z)=x^2+y^2+z^2-a^2=0$ ...(1)

$$\frac{\partial F}{\partial x}=2x;\ \frac{\partial F}{\partial y}=2y;\ \frac{\partial F}{\partial z}=2z$$

Thus, eq. of normal to surface given by eq (1) is

$$\frac{X-x}{\frac{\partial F}{\partial x}}=\frac{Y-y}{\frac{\partial F}{\partial y}}=\frac{Z-z}{\frac{\partial F}{\partial z}} \text{ i.e. } \frac{X-x}{x}=\frac{Y-y}{y}=\frac{Z-z}{z}$$

Q 63. Use method of Lagrange's to find the minimum value of $x^2+y^2+z^2$, given that $xyz=a^3$. **(PTU, Dec. 2009)**

Solution. Here $f(x, y, z)=x^2+y^2+z^2$

Let the given constraint be, $xyz-a^3=0$ ...(1)

$$f(x, y)=x^2+y^2+\frac{a^6}{x^2y^2}$$

i.e. $$\frac{\partial f}{\partial x}=f_x=2x-\frac{2a^6}{x^3y^2}$$ ...(2)

and $$f_y=2y-\frac{2a^6}{x^2y^3}$$ ...(3)

$$f_{xx}=2+\frac{6a^6}{x^4y^2}, f_{yy}=2+\frac{6a^6}{x^2y^4}, f_{xy}=\frac{4a^6}{x^3y^3}$$

For stationary values $f_x=0=f_y$

from (2), $$x^4=\frac{a^6}{y^2} \Rightarrow x^2=\pm\frac{a^3}{y}$$

from (3), $$2y=\frac{2a^6}{x^2y^3}$$ ...(3)

when $$x^2=\frac{a^3}{y}$$ from (3), $$y=\frac{a^6}{a^3y^2}$$

$\Rightarrow$ $y^3=a^3 \Rightarrow y=a$

$x^2=a^2 \Rightarrow x=\pm a$ i.e. $(a, a), (-a, a)$

Similarly stationary points are $(a, -a), (-a, -a)$

Case I. at (a, a), $A=8, B=4, C=8$

$$AC-B^2=64-16=48>0, A=8>0$$

$\therefore (a, a)$ is a point of Minima and $z=\frac{a^3}{a^2}=a$

$$\text{Min. value}=a^2+a^2+a^2=3a^2$$

Case II. at $(a, -a)$, $A=8, B=-4, C=8$

$$AC-B^2=64-16=48>0, A=8>0$$

$$\text{Min. Value}=x^2+y^2+z^2=a^2+a^2+a^2$$

Similarly at other two points yields Min. value $=3a^2$

Q 64. If $u=\tan^{-1}\frac{x^3+y^3}{x+y}$, prove that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\sin 2u$. **(PTU, Dec. 2009)**

Solution. Given, $$u=\tan^{-1}\frac{x^3\left(1+\left(\frac{y}{x}\right)^3\right)}{x\left(1+\frac{y}{x}\right)}$$

$$\tan u=x^2\phi\left(\frac{y}{x}\right)$$ $\therefore$ tan u is homo. function of deg. 2 in x & y

Hence By Euler's theorem we have

$$x\frac{\partial}{\partial x}(\tan u)+y\frac{\partial}{\partial y}(\tan u)=2\tan u$$

$$x\sec^2u\frac{\partial u}{\partial x}+y\sec^2u\frac{\partial u}{\partial y}=2\tan u \Rightarrow x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=2\sin u\cos u=\sin 2u$$ ...(1)

Q 65. If $xyz=8$, find the values of x, y for which $u=\frac{5xyz}{(x+2y+4z)}$ is a maximum. **(PTU, Dec. 2010)**

Solution. Given, $$f(x, y)=\frac{5\times 8}{x+2y+4\times\frac{8}{xy}}=\frac{40}{x+2y+\frac{32}{xy}}=\frac{40xy}{x^2y+2xy^2+32}$$

$$f_x=\frac{40y\left(32-x^2y\right)}{\left[x^2y+2xy^2+32\right]^2}, f_y=\frac{40x\left(32-2xy^2\right)}{\left[x^2y+2xy^2+32\right]^2}$$

for extreme values, $f_x=f_y=0$

$\Rightarrow$ $x^2y=32$ ...(1)

and $xy^2=16$ ...(2)

Q 60. Prove that the rectangular solid of maximum volume that can be inscribed in a sphere is a cube.

(PTU, May 2010, 2009)

Solution. Let x, y, z be the three dimensions of the rectangular solid volume of solid V = xyz

further the diagonal of solid must pass through the centre of sphere

diagonal of solid = diameter of sphere = d

$$\Rightarrow \quad \sqrt{x^2+y^2+z^2} = d \Rightarrow z = \sqrt{d^2-x^2-y^2}$$

$$V = xy\sqrt{d^2-x^2-y^2}$$

To maximise or minimise V it is convenient to maximise or minimise V²

i.e.
$$\rho = V^2 = x^2y^2(d^2-x^2-y^2)$$

$$\frac{\partial\rho}{\partial x} = y^2(2x\,d^2-4x^3-2xy^2) = xy^2(2d^2-4x^2-2y^2)$$

$$\frac{\partial\rho}{\partial y} = x^2(2d^2y-2x^2y-4y^3) = x^2y(2d^2-2x^2-4y^2)$$

For maximum or minimum $\frac{\partial\rho}{\partial y} = 0 = \frac{\partial\rho}{\partial x}$

$\Rightarrow \quad 2xy^2(d^2-2x^2-y^2) = 0$ ...(1)

and $2x^2y(d^2-x^2-2y^2) = 0$ ...(2)

on solving (1) and (2), we get

$$2x^2+y^2 = d^2 = x^2+2y^2 \Rightarrow x^2-y^2 = 0 \Rightarrow x = \pm y$$

eq (1) gives, $d^2 = 2x^2+x^2 \Rightarrow 3x^2 = d^2 \Rightarrow x = \frac{\pm d}{\sqrt{3}}$

$$\text{again } z = \sqrt{d^2-x^2-y^2} = \sqrt{d^2-\frac{d^2}{3}-\frac{d^2}{3}} = \frac{d}{\sqrt{3}}$$

point of maxima or minima is $\left(\frac{d}{\sqrt{3}}, \frac{d}{\sqrt{3}}, \frac{d}{\sqrt{3}}\right)$ [∵ x, y, z > 0]

$$\text{again } \frac{\partial^2\rho}{\partial x^2} = y^2(2d^2-12x^2-2y^2);\ \frac{\partial^2\rho}{\partial y^2} = x^2(2d^2-2x^2-12y^2)$$

$$\frac{\partial^2\rho}{\partial x\partial y} = (4xyd^2-8x^3y-8xy^3)$$

$$A = \frac{\partial^2\rho}{\partial x^2} \text{ at } \left(\frac{d}{\sqrt{3}}, \frac{d}{\sqrt{3}}\right) = \frac{d^2}{3}\left(2d^2-4d^2-\frac{2d^2}{3}\right) = \frac{-8d^4}{9} < 0$$

$$B = \frac{\partial^2\rho}{\partial x\partial y} \text{ at } \left(\frac{d}{\sqrt{3}}, \frac{d}{\sqrt{3}}\right) = \left(\frac{4d^4}{3}-\frac{8d^4}{9}-\frac{8d^4}{9}\right) = \frac{-4d^4}{9}$$

$$C = \frac{\partial^2\rho}{\partial y^2} \text{ at } \left(\frac{d}{\sqrt{3}}, \frac{d}{\sqrt{3}}\right) = \frac{d^2}{3}\left(2d^2-\frac{2d^2}{3}-4d^2\right) = \frac{-8d^4}{9}$$

$$AC - B^2 = \frac{64d^8}{81} - \frac{16d^8}{81} > 0 \text{ and } A < 0$$

$\left(\frac{d}{\sqrt{3}}, \frac{d}{\sqrt{3}}, \frac{d}{\sqrt{3}}\right)$ is a point of maxima

Since $x = y = z = \frac{d}{\sqrt{3}}$ rectangular solid inscribed in sphere is a cube.

Q 61. If $u = a^3x^2+b^3y^2+c^3z^2$ where $\frac{1}{x}+\frac{1}{y}+\frac{1}{z} = 1$, show that the stationary value of u is given by, $x = \frac{\Sigma a}{a}, y = \frac{\Sigma a}{b}, z = \frac{\Sigma a}{c}$.

(PTU, Dec. 2008)

Solution. Let us form $F(x, y, z) = a^3x^2+b^3y^2+c^3z^2+\lambda\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}-1\right)$

where, given constraint is $\frac{1}{x}+\frac{1}{y}+\frac{1}{z} = 1$ ...(*)

and $f(x, y, z) = a^3x^2+b^3y^2+c^3z^2$

For extreme values, we have

$$\frac{\partial F}{\partial x} = 2a^3x+\lambda\left(\frac{-1}{x^2}\right) = 0 \quad \text{...(1)}$$

$$\frac{\partial F}{\partial y} = 2b^3y+\lambda\left(\frac{-1}{y^2}\right) = 0 \quad \text{...(2)}$$

$$\frac{\partial F}{\partial z} = 2c^3z+\lambda\left(\frac{-1}{z^2}\right) = 0 \quad \text{...(3)}$$

from (1), we have $a^3x^3 = \frac{\lambda}{2} = b^3y^3 = c^3z^3$

$$\Rightarrow \quad ax = by = cz = \left(\frac{\lambda}{2}\right)^{\frac{1}{3}} = K$$

$$\Rightarrow \quad x = \frac{K}{a}, y = \frac{K}{b}, z = \frac{K}{c}$$

we get $x^2 = 2 - x \Rightarrow x^2 + x - 2 = 0$

$\Rightarrow (x-1)(x+2) = 0 \Rightarrow x = 1, -2$

$y = 1, 4$

point of intersections are (1, 1), (−2, 4)

For change of order of integration we divide the region of integration into horizontal strips. The region consists of two regions

$R'_2 = \{(x, y) ; 0 \le x \le 2 - y ; 1 \le y \le 2\}$

$R''_2 = \{(x, y) ; 0 \le x \le \sqrt{y} ; 0 \le y \le 1\}$ and $R_2 = R'_2 \cup R''_2$

$$\iint_{R_2} xy\,dy\,dx = \iint_{R'_2} xy\,dy\,dx + \iint_{R''_2} xy\,dy\,dx$$

$$= \int_1^2 \int_0^{2-y} xy\,dx\,dy + \int_0^1 \int_0^{\sqrt{y}} xy\,dx\,dy$$

$$= \int_1^2 \frac{y(2-y)^2\,dy}{2} + \int_0^1 \frac{y^2}{2}\,dy = \frac{1}{2}\left[\int_1^2 y\left(4 + y^2 - 4y\right)dy\right] + \frac{1}{6}$$

$$= \frac{1}{6} + \frac{1}{2}\left[2y^2 + \frac{y^4}{4} - \frac{4y^3}{3}\right]_1^2$$

$$= \frac{1}{6} + \frac{1}{2}\left[8 + 4 - \frac{32}{3} - 2 - \frac{1}{4} + \frac{4}{3}\right] = \frac{1}{6} + \frac{1}{2}\left[\frac{2}{3} - \frac{1}{4}\right]$$

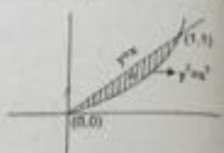
$$= \frac{3}{8}$$

Q 80. Using double integration, find the area enclosed by the curves, $y^2 = x^3$ and $y = x$. **(PTU, May 2009 ; Dec. 2005)**

Ans. Both curves $y^2 = x^3$ and $y = x$ intersects when $x^2 = x^3 \Rightarrow x^2(x-1) = 0 \Rightarrow x = 0, 1$

i.e. (0, 0), (1, 1) be the point of intersection.

$R = \{(x, y) ; x^{3/2} \le y \le x ; 0 \le x \le 1\}$



$$\text{Req. area} = \int_0^1 \int_{x^{3/2}}^{x} dy\,dx$$

$$= \int_0^1 \left[x - x^{3/2}\right]dx = \left[\frac{x^2}{2} - \frac{2x^{5/2}}{5}\right]_0^1$$

$$= \frac{1}{2} - \frac{2}{5} = \frac{1}{10} \text{ sq. units.}$$

Q 81. The region bounded by the curve $y = x^2+1$ and the line $y = -x+3$ is revolved about the x-axis to generate a solid. Find the volume of the solid. **(PTU, Dec. 2004)**

Ans. Both curves intersect at

$\Rightarrow x^2 = 3 - x - 1$

$\Rightarrow x^2 + x - 2 = 0$

$x = 1, -2, y = 2, 5$

(1, 2), (−2, 5) are the point of intersection

Volume of solid of revolution $= 2\pi \iint_R y\,dy\,dx$

$R = \{(x, y) : -2 \le x \le 1 ; x^2 + 1 \le y \le 3 - x\}$

$$= 2\pi \int_{-2}^{1} \int_{x^2+1}^{3-x} y\,dy\,dx$$

$$= \frac{2\pi}{2} \int_{-2}^{1} \left[(3-x)^2 - \left(x^2+1\right)^2\right]dx$$

$$\text{Req. Volume} = \pi \int_{-2}^{1} \left[-x^2 - x^4 - 6x + 8\right]dx$$

$$= \pi\left[\frac{-x^3}{3} - \frac{x^5}{5} - 3x^2 + 8x\right]_{-2}^{1}$$

$$= \pi\left[\frac{-1}{3} - \frac{1}{5} - 3 + 8 - \frac{8}{3} - \frac{32}{5} + 12 + 16\right] = \frac{117\pi}{5}$$

Q 82. Using the transformation $x+y = u$, $y = uv$, show that

$$\iint xy(1-x-y)^{1/2}\,dx\,dy = \frac{2\pi}{105}$$

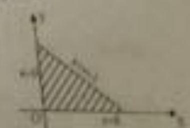
integration being taken over the area of the triangle bounded by the lines $x = 0$, $y = 0$, $x+y = 1$. **(PTU, Dec. 2004)**

Ans. The given transformation $x+y = u$, $y = uv \Rightarrow x = u - uv$, $y = uv$

$$\text{Now } J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1-v & -u \\ v & u \end{vmatrix}$$

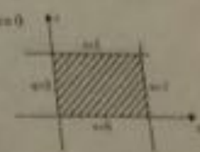
$$= u(1-v) + uv = u$$

$$dx\,dy = |J|\,du\,dv = u\,du\,dv$$



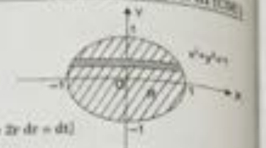
Now upper limit is given by putting $x+y = 1 \Rightarrow u = 1$

when $x = 0$, $u - uv = 0 \Rightarrow u = 0$, $v = 1$ and $y = 0$, $uv = 0 \Rightarrow u = 0$, $v = 0$



$$\therefore \text{ given Integral} = \int_0^1 \int_0^1 (u-uv)^{1/2}(uv)^{1/2}(1-u)^{1/2}\,u\,du\,dv$$

$$= \int_0^1 \int_0^1 u^2(1-v)^{1/2}(1-u)^{1/2}v^{1/2}\,du\,dv$$



given integral $= \int_0^1 \int_0^{2\pi} \log(r^2+1)\, r\, dr\, d\theta$

$= \int_0^{2\pi} d\theta \int_0^1 \log(r^2+1)\, r\, dr$

$= 2\pi \int_0^1 \log(t+1) \frac{dt}{2}$ [put $r^2 = t \Rightarrow 2r\, dr = dt$]

$= \pi \left[t \log(t+1) \Big|_0^1 - \int_0^1 \frac{t}{t+1} dt \right]$

$= \pi [\log 2] - \pi [t - \log(t+1)]_0^1$

$= \pi \log 2 - \pi (1 - \log 2)$

$= -\pi + 2\pi \log 2$

Q 76. Find $\iiint \frac{dx\, dy\, dz}{\sqrt{1-x^2-y^2-z^2}}$ where integration is taken over sphere $x^2 + y^2 + z^2 = 1$ in positive octant. **(PTU, May 2008, 2006)**

Ans. Changing into spherical co-ordinates by putting $x = r \sin\theta \cos\phi$; $y = r \sin\theta \sin\phi$, $z = r \cos\theta$ we get $r^2 = 1$ or $r = 1$

Since, it is the case of positive octant ∴ $R = \{r, \theta, \phi : 0 \le r \le 1 ; 0 \le \theta \le \frac{\pi}{2} ; 0 \le \phi \le \frac{\pi}{2}\}$

given integral $= \int_0^{\pi/2} d\phi \int_0^{\pi/2} \sin\theta\, d\theta \int_0^1 \frac{r^2}{\sqrt{1-r^2}} dr$ ($dx\, dy\, dz = r^2 \sin\theta\, dr\, d\theta\, d\phi$)

$= \int_0^{\pi/2} d\phi \int_0^{\pi/2} \sin\theta\, d\theta \int_0^1 \left(\frac{1}{\sqrt{1-r^2}} - \sqrt{1-r^2} \right) dr$

$= \left(\frac{\pi}{2} - 0\right) (-\cos\theta - 0)_0^{\pi/2} \left[\sin^{-1} r - \frac{r\sqrt{1-r^2}}{2} - \frac{1}{2}\sin^{-1} r \right]_0^1$

$= \left(\frac{\pi}{2} - 0\right) (-\cos\theta - 0)_0^{\pi/2} \left[\frac{\pi}{2} - 0 - \frac{1}{2}\frac{\pi}{2} - (0 - 0 - 0) \right]$

$= \frac{\pi}{2} (1) \left(\frac{\pi}{4}\right) = \frac{\pi^2}{8}$

Q 77. Find the volume generated by revolution of cardiod $r = a(1 - \cos\theta)$ about x-axis. **(PTU, May 2008 ; Dec. 2007 ; June 2007)**

Ans. Here $R = \{(r, \theta) : 0 \le r \le a(1 - \cos\theta) ; 0 \le \theta \le \pi\}$

Required volume $= 2\pi \int_0^{\pi} \int_0^{a(1-\cos\theta)} r^2 \sin\theta\, dr\, d\theta$

$= 2\pi \int_0^{\pi} \frac{a^3}{3} (1 - \cos\theta)^3 \sin\theta\, d\theta$

$= \frac{2\pi a^3}{3} \int_0^{\pi} (\sin\theta)(1 - \cos\theta)^3 d\theta$

$= \frac{2\pi a^3}{3} \left[\frac{(1-\cos\theta)^4}{4} \right]_0^{\pi} = \frac{2\pi a^3}{12} [0 + 2^4] = \frac{8\pi a^3}{3}$

Q 78. Find the volume of ellipsoid using triple integral. **(PTU, Dec. 2007 ; June 2007)**

Ans. Let the ellipsoid be $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ ...(1)

put

$x = aX$; $y = bY$, $z = cZ$

$dx = a dX$; $dy = b dY$; $dz = c dZ$

$dx dy dz = abc\, dX\, dY\, dZ$

and eq (1) becomes $X^2 + Y^2 + Z^2 = 1$

i.e. Region $V = \{X, Y, Z) ; X^2 + Y^2 + Z^2 \le 1\}$

Changing to spherical polar coordinates by the transformation

$X = r \sin\theta \cos\phi$, $Y = r \sin\theta \sin\phi$; $Z = r \cos\theta$

Region $V = \{r, \theta, \phi) ; 0 \le r \le 1 ; 0 \le \theta \le \pi ; 0 \le \phi \le 2\pi\}$

$dX dY dZ = r^2 \sin\theta\, dr\, d\theta\, d\phi$ and $X^2 + Y^2 + Z^2 = r^2$

$dx dy dz = abc\, r^2 \sin\theta\, dr d\theta d\phi$

Required volume $= \int_0^{2\pi} \int_0^{\pi} \int_0^1 abc\, r^2 \sin\theta\, dr\, d\theta\, d\phi$

$= \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta\, d\theta \int_0^1 abc\, r^2 dr = 2\pi\, (-\cos\theta)_0^{\pi}\, abc \left[\frac{r^3}{3}\right]_0^1$

$= \frac{4\pi}{3} abc$

Q 79. Change the order of integration in $\int_0^1 \int_{x^2}^{2-x} xy\, dx\, dy$ and hence evaluate. **(PTU, Dec. 2010 ; May 2009, 2006)**

Ans. Here we divide the region into vertical strip.

$R = \{(x, y) ; x^2 \le y \le 2 - x ; 0 \le x \le 1\}$

Now $x^2 = y$ and $y = 2 - x$ intersects

Q 68. Evaluate $\int_0^2\int_1^2\int_0^{yz} xyz\,dx\,dy\,dz$. (PTU, May 2009, 2006)

Ans. $\int_0^2\int_1^2\int_0^{yz} xyz\,dx\,dy\,dz = \int_0^2\int_1^2 yz\frac{x^2}{2}\Big]_0^{yz} dy\,dz = \int_0^2\int_1^2 \frac{y^3z^3}{2}dy\,dz$

$= \int_1^2 \frac{y^3}{2}dy\int_0^2 z^3 dz = \frac{y^4}{8}\Big]_1^2\left[\frac{z^4}{4}\right]_0^2 = 2\left[4-\frac{1}{4}\right] = \frac{15}{2}$

Q 69. Evaluate $\int_0^1\int_0^1 (x+2)\,dy\,dx$. (PTU, Dec. 2005)

Ans. $\int_0^1\int_0^1 (x+2)\,dy\,dx = \int_0^1 (x+2)(dx)\int_0^1 dy = \frac{(x+2)^2}{2}\Big]_0^1 y]_0^1$

$= \frac{1}{2}[9-4].1 = \frac{5}{2}$

Q 70. Evaluate : $\int_0^1\int_0^3 (x+5)\,dy\,dx$. (PTU, May 2005)

Ans. $\int_0^1\int_0^3 (x+5)\,dy\,dx = \int_0^1 (x+5)\,dx\int_0^3 dy = \frac{(x+5)^2}{2}\Big]_0^1 y]_0^3$

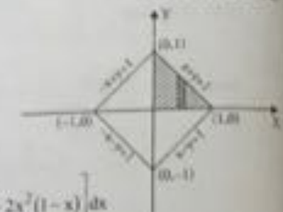
$= \frac{1}{2}[36-25].3 = \frac{33}{2}$.

Q 71. Sketch the region of integration and determine the order of intergration of the following integral

$\iint_R \left(y-2x^2\right)dx\,dy$.

where R is the region inside the square

$|x|+|y| = 1$. (PTU, Dec. 2004)



Ans. Given Integral $= 4\iint_{R'} \left(y-2x^2\right)dy\,dx$

$R' = \{(x,y) : 0 \le y \le 1-x, 0 \le x \le 1\}$

$= 4\int_0^1\int_0^{1-x} \left(y-2x^2\right)dy\,dx$

$= 4\int_0^1 \frac{y^2}{2} - 2x^2y\Big]_0^{1-x} dx = 4\int_0^1\left[\frac{(1-x)^2}{2} - 2x^2(1-x)\right]dx$

$= 4\left[\frac{(1-x)^3}{-6} - 2\left(\frac{x^3}{3} - \frac{x^4}{4}\right)\right]_0^1$

$= 4\left[-2\left(\frac{1}{3}-\frac{1}{4}\right) + \frac{1}{6} + 2(0)\right] = 0$

Q 72. Evaluate $\int_0^1\int_0^{y^2} e^{x/y}\,dx\,dy$. (PTU, May 2004)

Ans. $\int_0^1\int_0^{y^2} e^{x/y}\,dx\,dy = \int_0^1 ye^{x/y}\Big]_0^{y^2} dy = \int_0^1 y\left(e^y - 1\right)dy = \left[y\left(e^y - y\right) - \left(e^y - \frac{y^2}{2}\right)\right]_0^1$

$= (2(e^1-2) - (e^1-2)+1)$

$= (e^1-1)$

Q 73. Evalulate : $\int_0^2\int_0^{\sqrt{2x}} xy\,dy\,dx$. (PTU, Dec. 2003)

Ans. $\int_0^2\int_0^{\sqrt{2x}} xy\,dy\,dx = \int_0^2 x\frac{y^2}{2}\Big]_0^{\sqrt{2x}} dx = \int_0^2 \frac{x}{2}(2x)\,dx = \frac{x^3}{3}\Big]_0^2 = \frac{8}{3}$

Q 74. Write the limits of integration in $\iint_R xy\,dx\,dy$, **where R is the region inside the square** $|x|+|y|=1$. (PTU, May 2003)

Ans. The given region be $|x|+|y|=1$ ....(1)

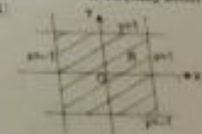
It meets x-axis *i.e.* y = 0 (1) gives $|x| = 1 \Rightarrow x = \pm 1$

i.e. point of intersection on x-axis are $(\pm 1, 0)$

Also it meets y-axis *i.e.* x = 0 *i.e.* $|y| = 1 \Rightarrow y = \pm 1$

point of intersection on y-axis are $(0, \pm 1)$

$\therefore R\{(x,y) ; -1 \le x \le 1, -1 \le y \le 1\}$



Q 75. Evaluate : $\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{+\sqrt{1-y^2}} \log_e(x^2+y^2+1)\,dx\,dy$ **by changing to polar co-ordinates.** (PTU, May 2003)

Ans. The region of integration in cartesian coordinates is given by

$R = \left\{(x,y) ; -\sqrt{1-y^2} \le x \le \sqrt{1-y^2} ; -1 \le y \le 1\right\}$

By changing to polar coordinates

$x = r\cos\theta ; y = r\sin\theta$

and $dx\,dy = r\,dr\,d\theta$

$R = \{(r,\theta) ; 0 \le r \le 1 ; 0 \le \theta \le 2\pi\}$

Volume of the ellipsoid $V = \iiint_R dx\,dy\,dz$

Where R be the region bdd by given ellipsoid.

$$\text{put } x = aX\ ;\ y = bY\ ;\ z = cZ$$
$$dx = a\,dX\ ;\ dy = b\,dY\ ;\ dz = c\,dZ$$

eq (1) reduces to $X^2 + Y^2 + Z^2 = 1$

$$dx\,dy\,dz = abc\,dX\,dY\,dZ$$
$$R = \{(X, Y, Z) : X^2 + Y^2 + Z^2 \le 1\}$$

Changing to spherical polar coordinates

$$\text{put } X = r\sin\theta\cos\phi\ ;\ Y = r\sin\theta\sin\phi\ ;\ Z = r\cos\theta$$
$$X^2 + Y^2 + Z^2 = r^2$$

and $dX\,dY\,dZ = r^2\sin\theta\,dr\,d\theta\,d\phi$

$$\text{Req. volume } V = \iiint_R r^2\sin\theta\ abc\ dr\,d\theta\,d\phi$$

Here $R = \{(r, \theta, \phi) ; 0 \le r \le 1 ; 0 \le \theta \le \pi ; 0 \le \phi \le 2\pi\}$

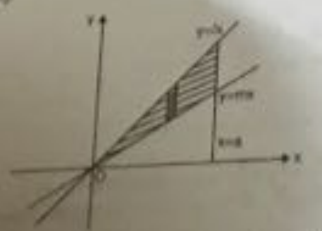
$$= \int_0^1\int_0^{\pi}\int_0^{2\pi} r^2\sin\theta\ abc\ d\phi\,d\theta\,dr$$
$$= \int_0^1 abc\,r^2\,dr \int_0^{\pi}\sin\theta\,d\theta \int_0^{2\pi} d\phi$$
$$= \frac{abc}{3}(+2)\,2\pi = \frac{4\pi}{3}abc$$

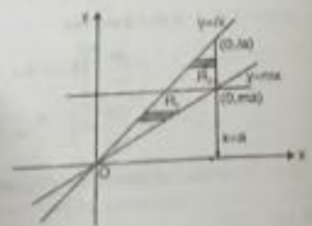
Q 91. Change the order of integration in, $\int_0^a\int_{mx}^{lx} f(x, y)\,dy\,dx$. (PTU, Dec. 2008)

Ans. The given region can be written as

$$R = \{(x, y) ; mx \le y \le lx ; 0 \le x \le a\}$$

Now we divide the region R into horizontal strips and R is dividing into two regions R_1 and R_2





where, $R_1 = \{(x, y) ; \frac{y}{l} \le x \le \frac{y}{m} ; 0 \le y \le ma\}$

and $R_2 = \{(x, y) ; \frac{y}{l} \le x \le a ; ma \le y \le la\}$

$$\text{Given Integral} = \iint_{R_1} f(x, y)\,dx\,dy - \iint_{R_2} f(x, y)\,dx\,dy$$
$$= \int_0^{ma}\int_{y/l}^{y/m} f(x, y)\,dx\,dy + \int_{ma}^{la}\int_{y/l}^{a} f(x, y)\,dx\,dy$$

Q 92. Evaluate $\int_0^1\int_0^{\sqrt{1-y^2}} (x^2 - y^2)\,dx\,dy$, by changing to polar co-ordinates. (PTU, May 2010)

Solution. Given region can be written as

$$R = \{(x, y) ; 0 \le x \le \sqrt{1-y^2}, 0 \le y \le 1\}$$

Changing to polar coordinates by the transformations

$$x = r\cos\theta ; y = r\sin\theta$$

and $dx\,dy = r\,dr\,d\theta$

s.t. $x^2 + y^2 = r^2$

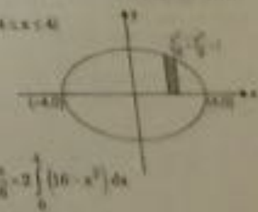
and $R = \{(r, \theta) ; 0 \le r \le 1 ; 0 \le \theta \le \frac{\pi}{2}\}$

$$\text{Given integral} = \int_0^{\pi/2}\int_0^1 r^2(r\,dr\,d\theta) = \int_0^{\pi/2} d\theta \int_0^1 r^3\,dr$$
$$= \frac{\pi}{2}\cdot\frac{1}{4} = \frac{\pi}{8}$$

Q 93. Find the volume generated by revolving the ellipse : $\frac{x^2}{16} + \frac{y^2}{9} = 1$ about the x-axis. (PTU, Dec. 2009)

Solution. Region $R = \{(x, y) : 0 \le y \le \frac{3}{4}\sqrt{16 - x^2} ; -4 \le x \le 4\}$

Required volume generated about x-axis



$$= 2\pi\int_{-4}^{4}\int_0^{\frac{3}{4}\sqrt{16-x^2}} y\,dy\,dx$$
$$= \frac{2\pi}{2}\int_{-4}^{4}\frac{9}{16}(16 - x^2)\,dx = \frac{9\pi}{16}\cdot 2\int_0^4 (16 - x^2)\,dx$$
$$= \frac{9\pi}{8}\left[16x - \frac{x^3}{3}\right]_0^4$$

$\Rightarrow \quad \sqrt{2ax} = \sqrt{2ax - x^2}$

$\Rightarrow \quad x = 0$

i.e. (0, 0)

again when $x = 2a$, $y = \pm 2a$

i.e. $x = 2a$ meets $y = \sqrt{2ax}$ in two points

$(2a, +2a)$ and $(2a, -2a)$

Here we divide the region into horizontal strips

$$y = \sqrt{2ax} \Rightarrow x = \frac{y^2}{2a}$$

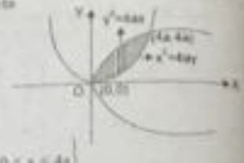
and $$y = \sqrt{2ax - x^2} \Rightarrow x^2 - 2ax + y^2 = 0 \Rightarrow x = \frac{2a \pm \sqrt{4a^2 - 4y^2}}{2}$$

Now $x = a + \sqrt{a^2 - y^2}$ corresponds to Ist quadrant

Region is given by $= \left\{ (x, y) : \frac{y^2}{2a} \le x \le a + \sqrt{a^2 - y^2} \text{ and } 0 \le y \le 2a \right\}$

Q 88. Find the area common to the parabolas $y^2 = 4ax$ and $x^2 = 4ay$ by using double integration. (PTU, Dec. 2003)

Ans. The given curves $y^2 = 4ax$ and $x^2 = 4ay$ intersects



we get, $\frac{x^4}{16a^2} = 4ax \Rightarrow x^4 = 64a^3x$

$\Rightarrow \quad x(x^3 - 64a^3) = 0$

$\Rightarrow \quad x = 0, 4a$

i.e. $\quad y = 0, 4a$

i.e. point of intersections are (0, 0) and (4a, 4a)

$$R = \left\{ (x, y) : \frac{x^2}{4a} \le y \le 2\sqrt{ax},\ 0 \le x \le 4a \right\}$$

$$\text{Req. area} = \int_0^{4a} \int_{x^2/4a}^{2\sqrt{ax}} dy\, dx$$

$$= \int_0^{4a} \left[2\sqrt{ax} - \frac{x^2}{4a} \right] dx = \left[\frac{2 \times 2\sqrt{a}}{3} x^{3/2} - \frac{x^3}{12a} \right]_0^{4a}$$

$$= \frac{4}{3}\sqrt{a} \cdot 8a^{3/2} - \frac{64a^3}{12a} = \frac{32}{3}a^2 - \frac{16a^2}{3} = \frac{16}{3}a^2.$$

Q 89. Find the volume of the region D enclosed by the surfaces $z = x^2 + 3y^2$ and $z = 8 - x^2 - y^2$. (PTU, May 2005)

Ans. The given surface is $z = x^2 + 3y^2$ and $z = 8 - x^2 - y^2$ intersect

When $x^2 + 3y^2 = 8 - x^2 - y^2 \Rightarrow 2x^2 + 4y^2 = 8 \Rightarrow x^2 + 2y^2 = 4$

i.e. Both surfaces intersects in ellipse $x^2 + 2y^2 = 4$

Region $V = \left\{ (x, y, z) ; x^2 + 3y^2 \le z \le 8 - x^2 - y^2 ; -\sqrt{\frac{4 - x^2}{2}} \le y \le \sqrt{\frac{4 - x^2}{2}} \text{ and } -2 \le x \le 2 \right\}$

$$\text{Req. volume} = \int_{-2}^{2} \int_{-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}} \int_{x^2+3y^2}^{8-x^2-y^2} dz\, dy\, dx$$

$$= \int_{-2}^{2} \int_{-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}} \left[8 - 2x^2 - 4y^2 \right] dy\, dx$$

$$= 2 \int_{-2}^{2} \left[(8 - 2x^2)y - \frac{4y^3}{3} \right]_0^{\sqrt{\frac{4-x^2}{2}}} dx$$

$$= 2 \int_{-2}^{2} \left[(8 - 2x^2)\sqrt{\frac{4 - x^2}{2}} - \frac{4}{3}\left(\frac{4 - x^2}{2} \right)^{3/2} \right] dx$$

$$= 2 \int_{-2}^{2} \sqrt{\frac{4 - x^2}{2}} \left[(8 - 2x^2) - \frac{2}{3}(4 - x^2) \right] dx$$

$$= \frac{2}{3} \int_{-2}^{2} \sqrt{\frac{4 - x^2}{2}} \left[24 - 6x^2 - 8 + 2x^2 \right] dx$$

$$= \frac{2}{3} \int_{-2}^{2} \sqrt{\frac{4 - x^2}{2}} (16 - 4x^2)\, dx = \frac{8}{3} \times \frac{2}{\sqrt{2}} \int_0^2 (4 - x^2)^{3/2} dx$$

put $x = 2\sin\theta \Rightarrow dx = 2\cos\theta\, d\theta$

$$= \frac{16}{3\sqrt{2}} \int_0^{\pi/2} 8\cos^3\theta \times 2\cos\theta\, d\theta = \frac{256}{3\sqrt{2}} \int_0^{\pi/2} \cos^4\theta\, d\theta$$

$$= \frac{256}{3\sqrt{2}} \times \frac{3 \cdot 1}{4 \cdot 2} \cdot \frac{\pi}{2} = \frac{16}{\sqrt{2}}\pi = 8\sqrt{2}\pi \text{ cubic units}$$

Q 90. Find the volume of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, using integration. (PTU, Dec. 2007)

Ans. The given ellipsoid is $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ ...(1)

$$= \int_0^1 u^2 (1-u)^{1/2} du \int_0^1 v^{1/2} (1-v)^{1/2} dv$$

put $u = \sin^2\theta \Rightarrow du = 2\sin\theta\cos\theta\, d\theta$ and $v = \sin^2\phi$ $dv = 2\sin\phi\cos\phi\, d\phi$

$$= \int_0^{\pi/2} 2\sin^5\theta\cos^2\theta\, d\theta \int_0^{\pi/2} 2\sin^2\phi\cos^2\phi\, d\phi$$

$$= 2\left[\frac{4.2.1}{7.5.3.1}\right] \times 2\left[\frac{1.1.\pi}{4.2.2}\right] = \frac{2\pi}{105}$$

Q 83. Find the volume common to the cylinders, $x^2 + y^2 = a^2$ and $x^2 + z^2 = a^2$.

(PTU, May 2004)

Ans. Here required volume $= 2\iint_R z\, dx\, dy$ where $z = \sqrt{a^2 - x^2}$

Here R be the region of intergration i.e. section of cylinder $x^2 + y^2 = a^2$ is a circle in XOY plane and it is symmetrical about four quadrants

i.e. Req. volume $= 2 \times 4 \iint_{R'} z\, dx\, dy$

Where $R' = \left\{(x, y) : 0 \le y \le \sqrt{a^2 - x^2} ; 0 \le x \le a\right\}$

$$= 8\int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{a^2 - x^2}\, dy\, dx = 8\int_0^a \left(a^2 - x^2\right) dx$$

$$= 8\left[a^2 x - \frac{x^3}{3}\right]_0^a = 8\left[a^3 - \frac{a^3}{3}\right] = \frac{16a^3}{3}.$$

Q 84. Evaluate : $\int_0^1 \int_x^1 \sin y^2\, dy\, dx$ by changing the order of integration.

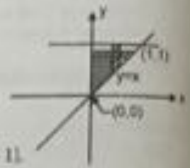
(PTU, May 2012)

Solution. Here given region is divided into vertical strips

i.e. $R = \{(x, y) : x \le y \le 1 ; 0 \le x \le 1\}$

For change of order of integration we divide the region into horizontal strips.

i.e. $R' = \{(x, y) : 0 \le x \le y ; 0 \le y \le 1\}$



Thus given Integral $= \int_0^1 \int_0^y \sin y^2\, dx\, dy = \int_0^1 y \sin y^2\, dy$

put $y^2 = t \Rightarrow 2y\, dy = dt$

$$= \int_0^1 \frac{1}{2} \sin t\, dt = \frac{1}{2}\left[-\cos t\right]_0^1 = -\frac{1}{2}\left[\cos 1 - 1\right].$$

Q 85. Evaluate : $\int_0^\infty \int_0^\infty e^{-\left(x^2+y^2\right)} dx\, dy$.

(PTU, Dec. 2007, 2003)

Ans. In the given region of integration x varies from 0 to ∞ and y varies from 0 to ∞ and changing to polar coordinates by the transformations $x = r\cos\theta$ $y = r\sin\theta$ and $dx\, dy = r\, dr\, d\theta$ and $x^2 + y^2 = r^2$ as x,y lies in 1st quadrant

$$R = \{(r, \theta) ; 0 \le r < \infty ; 0 \le \theta \le \frac{\pi}{2}\}$$

given Integral $= \int_0^{\pi/2} \int_0^\infty e^{-r^2} r\, dr\, d\theta$

$$= \int_0^{\pi/2} d\theta \int_0^\infty e^{-r^2} r\, dr = \frac{\pi}{2}\left[\frac{e^{-r^2}}{-2}\right]_0^\infty$$

given integral $= \frac{-\pi}{4}\left[e^{-\infty} - e^0\right] = \frac{\pi}{4}$

Q 86. Calculate the volume of the solid bounded by surfaces $x = 0, y = 0, z = 0$ and $x + y + z = 1$.

(PTU, Dec. 2003)

Ans. Here the region of integration is given by

$$V = \{(x, y, z) ; 0 \le z \le 1 - x - y ; 0 \le y \le 1 - x ; 0 \le x \le 1\}$$

Req. volume $= \int_0^1 \int_0^{1-x} \int_0^{1-x-y} dz\, dy\, dx = \int_0^1 \int_0^{1-x} (1 - x - y)\, dy\, dx$

$$= \int_0^1 \left[\frac{(1-x-y)^2}{-2}\right]_0^{1-x} dx = \frac{-1}{2}\int_0^1 \left[0 - (1-x)^2\right] dx$$

$$= \frac{1}{2}\int_0^1 (1-x)^2\, dx = \frac{1}{2}\left[\frac{(1-x)^3}{-3}\right]_0^1 = \frac{-1}{6}[0 - 1] = \frac{1}{6}$$

Q 87. Change the order of integration : $\int_0^{2a} \int_{\sqrt{2ax-x^2}}^{\sqrt{2ax}} V\, dy\, dx$. (PTU, Dec. 2003)

Ans. Here in the given region we divide into vertical strips

$$R = \left\{(x, y) : \sqrt{2ax - x^2} \le y \le \sqrt{2ax} ; 0 \le x \le 2a\right\}$$

Both curves $\sqrt{2ax - x^2} = y$

and $y = \sqrt{2ax}$

Q 99. Evaluate $\iint_D e^{-(x^2+y^2)}\,dx\,dy$ **, where D is the region bounded by** $x^2+y^2=a^2$ **by changing in polar coordinate.** (PTU, May 2011)

Solution. Changing the given region D into polar coordinates by the transformations $x = r\cos\theta$, $y = r\sin\theta$ and $dx\,dy = r\,dr\,d\theta$

i.e. $D = \{(r,\theta) : 0 \le r \le a,\ 0 \le \theta \le 2\pi\}$

$$\text{Given integral} = \int_0^{2\pi}\int_0^{a} e^{-r^2}\, r\,dr\,d\theta = \int_0^{2\pi} d\theta \int_0^{a} e^{-r^2}\, r\,dr$$

$$= 2\pi \cdot \frac{1}{2}\left[e^{-t}\right]_0^{a^2} \qquad \left[\text{where } r^2 = t,\ r\,dr = \frac{dt}{2}\right]$$

$$= \pi\left[e^{-a^2} - 1\right]$$

Q 100. Evaluate the integral $\int_0^1\int_x^{\sqrt{2-x^2}} \frac{x}{\sqrt{x^2+y^2}}\,dy\,dx$ **by changing the order of integration.** (PTU, May 2019)

Solution. Here we divide the region into vertical strips.

$$R_1 = \left\{(x,y) : x \le y \le \sqrt{2-x^2},\ 0 \le x \le 1\right\}$$

For change of order of integration. We divide the region into two regions R_1' & R_1'' (Horizontal strips)

$$R_1' = \{(x,y) : 0 \le x \le y,\ 0 \le y \le 1\}$$

$$R_1'' = \left\{(x,y) : 0 \le x \le \sqrt{2-y^2},\ 1 \le y \le \sqrt{2}\right\}$$

$$\int_0^1\int_x^{\sqrt{2-x^2}} \frac{x\,dy\,dx}{\sqrt{x^2+y^2}} = \int_0^1\int_0^y \frac{x\,dx\,dy}{\sqrt{x^2+y^2}} + \int_1^{\sqrt{2}}\int_0^{\sqrt{2-y^2}} \frac{x\,dx\,dy}{\sqrt{x^2+y^2}}$$

$$= \int_0^1\left[\sqrt{x^2+y^2}\right]_0^y dy + \int_1^{\sqrt{2}}\left[\sqrt{x^2+y^2}\right]_0^{\sqrt{2-y^2}} dy$$

$$= \int_0^1\left(\sqrt{2}y - y\right)dy + \int_1^{\sqrt{2}}\left(\sqrt{2} - y\right)dy = \left(\sqrt{2}-1\right)\frac{1}{2} - \frac{1}{2}\left[\left(\sqrt{2}-y\right)^2\right]_1^{\sqrt{2}}$$

$$= \left(\sqrt{2}-1\right)\frac{1}{2} - \frac{1}{2}\left[0 - \left(\sqrt{2}-1\right)^2\right] = \frac{\sqrt{2}-1}{2}\left[1+\sqrt{2}-1\right] = \frac{\sqrt{2}-1}{\sqrt{2}} = 1 - \frac{1}{\sqrt{2}}$$

Q 101. Show that the limit for the function $f(x,y) = \frac{x^2+y^2}{x^2-y^2}$ **does not exists as** $(x,y) \to (0,0)$. (PTU, Dec. 2020)

Ans. Let $(x,y) \to (0,0)$ along the curve $y = mx$ as $y \to 0 \Rightarrow x \to 0$

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{x^2-y^2} = \lim_{x\to 0} \frac{x^2(1+m^2)}{x^2(1-m^2)}$$

$$= \frac{1+m^2}{1-m^2}$$

which is not unique, for different values of m, given limit has different values.

So $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exists.

Q 102. Evaluate the integral $\int_{-1}^{1}\int_0^{z}\int_{x-z}^{x+z} dy\,dx\,dz$. (PTU, Dec. 2020)

Ans. $\int_{-1}^{1}\int_0^{z}\left[\int_{x-z}^{x+z}(x+y+z)\,dy\right]dx\,dz$

$$= \int_{-1}^{1}\int_0^{z}\left[xy + \frac{y^2}{2} + zy\right]_{x-z}^{x+z} dx\,dz$$

$$= \int_{-1}^{1}\int_0^{z}\left[x(x+z) + \frac{(x+z)^2}{2} + z(x+z) - x(x-z) - \frac{(x-z)^2}{2} - z(x-z)\right]dx\,dz$$

$$= \int_{-1}^{1}\int_0^{z}\left[x^2 + xz + \frac{(x+z)^2}{2} + zx + z^2 - x^2 + xz - \frac{(x-z)^2}{2} - zx + z^2\right]dx\,dz$$

$$= \int_{-1}^{1}\int_0^{z}\left(4xz + 2z^2\right)dx\,dz = \int_{-1}^{1}\left[2x^2z + 2z^2x\right]_0^{z} dz$$

$$= \int_{-1}^{1} 4z^3\,dz = 0 \qquad [\because f(z) = 4z^3 \text{ is an odd function}]$$

Q 103. Write down the Taylor's series expansion for sin x about $x = \frac{\pi}{2}$. (PTU, Dec. 2020)

Ans. Given $f(x) = \sin x$; $f\left(\frac{\pi}{2}\right) = 1$

$f'(x) = \cos x$; $f'\left(\frac{\pi}{2}\right) = 0$

Q 99. Evaluate $\iint_D e^{-(x^2+y^2)}\,dx\,dy$, **where D is the region bounded by** $x^2+y^2=a^2$ **by changing in polar coordinate.** (PTU, May 2011)

Solution. Changing the given region D into polar coordinates by the transformations $x = r\cos\theta$; $y = r\sin\theta$ and $dx\,dy = r\,dr\,d\theta$

i.e. $D = \{(r,\theta) : 0 \le r \le a ; 0 \le \theta \le 2\pi\}$

$$\text{Given integral} = \int_0^{2\pi}\int_0^{a} e^{-r^2}\,r\,dr\,d\theta = \int_0^{2\pi} d\theta \int_0^{a} e^{-r^2}\,r\,dr$$

$$= 2\pi \cdot \frac{1}{2}\left[e^{-t}\right]_0^{a^2} \qquad \left[\text{where } r^2 = t,\ r\,dr = \frac{dt}{2}\right]$$

$$= \pi\left[e^{-a^2} - 1\right]^0$$

Q 100. Evaluate the integral $\int_0^1\int_x^{\sqrt{2-x^2}} \frac{x}{\sqrt{x^2+y^2}}\,dy\,dx$ **by changing the order of integration.** (PTU, May 2016)

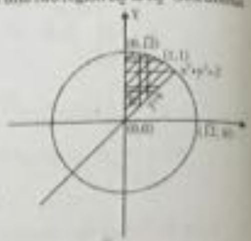
Solution. Here we divide the region into vertical strips

$$R_1 = \{(x,y) : x \le y \le \sqrt{2-x^2} ; 0 \le x \le 1\}$$

For change of order of integration. We divide the region into two region R_2' & R_2'' (Horizontal Strips)

$$R_2' = \{(x,y) : 0 \le x \le y ; 0 \le y \le 1\}$$

$$R_2'' = \{(x,y) : 0 \le x \le \sqrt{2-y^2} ; 1 \le y \le \sqrt{2}\}$$



$$\int_0^1\int_x^{\sqrt{2-x^2}} \frac{x\,dy\,dx}{\sqrt{x^2+y^2}} = \int_0^1\int_0^y \frac{x\,dx\,dy}{\sqrt{x^2+y^2}} + \int_1^{\sqrt{2}}\int_0^{\sqrt{2-y^2}} \frac{x\,dx\,dy}{\sqrt{x^2+y^2}}$$

$$= \int_0^1\left[\sqrt{x^2+y^2}\right]_0^y dy + \int_1^{\sqrt{2}}\left[\sqrt{x^2+y^2}\right]_0^{\sqrt{2-y^2}} dy$$

$$= \int_0^1\left(\sqrt{2}y - y\right)dy + \int_1^{\sqrt{2}}\left(\sqrt{2} - y\right)dy = \left(\sqrt{2}-1\right)\frac{1}{2} - \frac{1}{2}\left[\left(\sqrt{2}-y\right)^2\right]_1^{\sqrt{2}}$$

$$= \left(\sqrt{2}-1\right)\frac{1}{2} - \frac{1}{2}\left[0 - \left(\sqrt{2}-1\right)^2\right] = \frac{\sqrt{2}-1}{2}\left[1+\sqrt{2}-1\right] = \frac{\sqrt{2}-1}{\sqrt{2}} = 1 - \frac{1}{\sqrt{2}}$$

Q 101. Show that the limit for the function $f(x,y) = \frac{x^2+y^2}{x^2-y^2}$ **does not exists as** $(x,y) \to (0,0)$. (PTU, Dec. 2020)

Ans. Let $(x, y) \to (0,0)$ along the curve $y = mx$ as $y \to 0 \Rightarrow x \to 0$

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{x^2-y^2} = \lim_{x\to 0}\frac{x^2(1+m^2)}{x^2(1-m^2)}$$

$$= \frac{1+m^2}{1-m^2}$$

which is not unique, for different values of m, given limit has different values

So $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exists.

Q 102. Evaluate the integral $\int_{-1}^{1}\int_0^{z}\int_{x-z}^{x+z} dy\,dx\,dz$. (PTU, Dec. 2020)

Ans. $\int_{-1}^{1}\int_0^{z}\left[\int_{x-z}^{x+z}(x+y+z)\,dy\right]dx\,dz$

$$= \int_{-1}^{1}\int_0^{z}\left[xy + \frac{y^2}{2} + zy\right]_{x-z}^{x+z} dx\,dz$$

$$= \int_{-1}^{1}\int_0^{z}\left[x(x+z) + \frac{(x+z)^2}{2} + z(x+z) - x(x-z) - \frac{(x-z)^2}{2} - z(x-z)\right]dx\,dz$$

$$= \int_{-1}^{1}\int_0^{z}\left[x^2 + xz + \frac{(x+z)^2}{2} + zx + z^2 - x^2 + xz - \frac{(x-z)^2}{2} - zx + z^2\right]dx\,dz$$

$$= \int_{-1}^{1}\int_0^{z}\left[(4xz + 2z^2)\,dx\right]dz = \int_{-1}^{1}\left[2x^2z + 2z^2x\right]_0^z dz$$

$$= \int_{-1}^{1} 4z^3\,dz = 0 \qquad [\because f(z) = 4z^3 \text{ is an odd function}]$$

Q 103. Write down the Taylor's series expansion for sin x about $x = \frac{\pi}{2}$. (PTU, Dec. 2020)

Ans. Given $f(x) = \sin x$; $f\left(\frac{\pi}{2}\right) = 1$

$f'(x) = \cos x$; $f'\left(\frac{\pi}{2}\right) = 0$

$= \frac{8\pi}{[illegible]}\left[64 - \frac{64}{4}\right] = \frac{8\pi}{[illegible]} \cdot \frac{128}{3} = [illegible]\pi$

Q 94. Change the order of integration in $I = \int_0^{4a}\int_{\frac{x^2}{4a}}^{2\sqrt{ax}} dy\, dx$ and hence evaluate it. (PTU, Dec. 2009)

Solution. Here in given region, we divide the region into vertical strips

i.e. $R = \{(x, y) : \frac{x^2}{4a} \le y \le 2\sqrt{ax}\ ;\ 0 \le x \le 4a\}$

Both curves $x^2 = 4ay$ and $y^2 = 4ax$

intersects when $\left(\frac{x^2}{4a}\right)^2 = 4ax$

$\Rightarrow \quad x(x^3 - 64a^3) = 0$

$\Rightarrow \quad x = 0, 4a \quad y = 0, 4a$

i.e. points of intersection are (0, 0) and (4a, 4a)

For change of order of integration, we divide the region into horizontal strips

i.e. $R = \{(x, y) : \frac{y^2}{4a} \le x \le 2\sqrt{ay}\ ;\ 0 \le y \le 4a\}$

$$I = \int_0^{4a}\int_{\frac{y^2}{4a}}^{2\sqrt{ay}} dx\, dy = \int_0^{4a}\left[2\sqrt{ay} - \frac{y^2}{4a}\right] dy$$

$$= \left[\frac{4}{3}\sqrt{a}\, y^{3/2} - \frac{y^3}{12a}\right]_0^{4a} = \frac{32}{3}a^2 - \frac{16}{3}a^2 = \frac{16}{3}a^2$$

Q 95. Find the volume of the tetrahedron bounded by the coordinate axes and the plane $x + y + z = a$ by triple integration. (PTU, Dec. 2009)

Solution. Here, Region $V = \{(x, y, z) ; 0 \le z \le a - y - x ; 0 \le y \le a - x ; 0 \le x \le a\}$

$$\text{Required volume} = \int_0^a\int_0^{a-x}\int_0^{a-y-x} dz\, dy\, dx = \int_0^a\int_0^{a-x}(a - y - x)\, dy\, dx$$

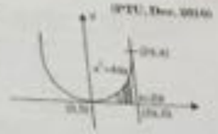
$$= \int_0^a \left[\frac{(a-y-x)^2}{-2}\right]_0^{a-x} dx = -\frac{1}{2}\int_0^a \left[0 - (a-x)^2\right] dx$$

$$= \frac{1}{2}\left[\frac{(a-x)^3}{-3}\right]_0^a = -\frac{1}{6}[0 - a^3] = \frac{a^3}{6}$$

Q 96. Evaluate $\iint_A xy\, dx\, dy$, where A is the domain bounded by x-axis, ordinate $x = 2a$ and the curve $x^2 = 4ay$. (PTU, Dec. 2010)

Solution. Here region of integration is given by

$R = \left\{(x, y) : 0 \le y \le \frac{x^2}{4a}, 0 \le x \le 2a\right\}$



$$\iint_R xy\, dy\, dx = \int_0^{2a}\int_0^{x^2/4a} xy\, dy\, dx$$

$$= \int_0^{2a}\left[\frac{xy^2}{2}\right]_0^{x^2/4a} = \int_0^{2a}\frac{x}{2}\left[\frac{x^4}{16a^2}\right] dx = \frac{1}{32a^2}\left[\frac{x^6}{6}\right]_0^{2a} = \frac{a^4}{3}$$

Q 97. Evaluate $\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} xyz\, dz\, dy\, dx$. (PTU, Dec. 2010)

Solution. Given Integral

$$= \int_0^1\int_0^{\sqrt{1-x^2}} xy\left(\frac{z^2}{2}\right)_0^{\sqrt{1-x^2-y^2}} dy\, dx = \int_0^1\int_0^{\sqrt{1-x^2}} \frac{xy}{2}(1 - x^2 - y^2)\, dy\, dx$$

$$= \frac{1}{2}\int_0^1 x\left[\int_0^{\sqrt{1-x^2}} y(1 - x^2 - y^2)\, dy\right] dx = -\frac{1}{4}\int_0^1 x\left[\frac{(1-x^2-y^2)^2}{2}\right]_0^{\sqrt{1-x^2}} dx$$

$$= -\frac{1}{8}\int_0^1 x\left\{0 - (1-x^2)^2\right\} dx = \frac{1}{8}\int_0^1 x(x^4 - 2x^2 + 1)\, dx = \frac{1}{8}\int_0^1 (x^5 - 2x^3 + x)\, dx$$

$$= \frac{1}{8}\left(\frac{x^6}{6} - \frac{2x^4}{4} + \frac{x^2}{2}\right)_0^1 = \frac{1}{8}\left(\frac{1}{6} - \frac{2}{4} + \frac{1}{2} - 0\right) = \frac{1}{48}$$

Q 98. Evaluate the integral $\int_1^2\int_0^x \frac{dx\, dy}{x^2 + y^2}$. (PTU, May 2011)

Solution. $\int_1^2\int_0^x \frac{dx\, dy}{x^2+y^2} = \int_1^2\left[\int_0^x \frac{1}{x^2+y^2} dy\right] dx$

$$= \int_1^2 \frac{1}{x}\left[\tan^{-1}\left(\frac{y}{x}\right)\right]_0^x dx = \int_1^2 \frac{1}{x}\frac{\pi}{4} dx = \frac{\pi}{4}[\log x]_1^2 = \frac{\pi}{4}\log 2$$

(ii) If $\lim_{n\to\infty} \frac{u_n}{v_n} = 0$ then Σu_n cgs if Σv_n cgs

(iii) If $\lim_{n\to\infty} \frac{u_n}{v_n} = \infty$ then Σu_n dgs if Σv_n dgs

Auxillary Series : The series $\Sigma \frac{1}{n^p}$ cgs if $p > 1$ and diverges if $p \le 1$

D' Alembert Ratio Test : If Σu_n be +ve term series and $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = l$ then Σu_n cgs if $l < 1$ and dgs if $l > 1$ and at $l = 1$, test fails

Cauchy's root test : If $\lim_{n\to\infty} (u_n)^{\frac{1}{n}} = l$ Then the +ve term series Σu_n is cgs if $l < 1$ and diverges if $l > 1$ and at $l = 1$, test fails

Raabe's Test : If $\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = l$ Then the +ve term series Σu_n is converges if $l > 1$ and dgs if $l < 1$ and at $l = 1$, test fails.
This test is applicable when ratio test fails

Logarithmic Test : If $\lim_{n\to\infty} n \log \frac{u_n}{u_{n+1}} = l$ then the +ve term series Σu_n converges if $l > 1$ and dgs if $l < 1$ and test fails at $l = 1$.

This test is convenient to apply when $\frac{u_n}{u_{n+1}}$ contains e.

Gauss's Test : If Σu_n be positive term series and $\frac{u_n}{u_{n+1}} = 1 + \frac{\mu}{n} + O\left(\frac{1}{n^2}\right)$ then the series Σu_n converges if $\mu > 1$ and diverges if $\mu \le 1$.

Cauchy's Integral Test : If f be defined non -ve and decreasing $\forall x \ge 1$ then the series $\sum_{n=1}^{\infty} f(n)$ and $\int_1^{\infty} f(x)\,dx$ behave alike.

Alternating Series Test : A series is said to be alternating series if all the terms of the series are alternatively +ve or -ve. If the seq $\{u_n\}$ is monotonically decreasing sequence and $\lim_{n\to\infty} u_n = 0$ Then the alternating series $\Sigma (-1)^{n-1} u_n$ converges.

Note : If $u_n \to a \neq 0$ Then the series $\Sigma (-1)^{n-1} u_n$ oscillates finitely

Weierstrass's M-Test : A series $\Sigma u_n(x)$ cgs uniformaly and absolutely if $\exists$ a convergent series ΣM_n of the constants s.t. $|u_n(x)| \le M_n \;\forall\; n \in N$

Jasmeet Singh

QUESTION-ANSWERS

Q 1. Explain the convergence and divergence of a series. (PTU, Dec. 2007)

Solution. Let $<a_n>$ be a real sequence. The sum of first n terms namely $a_1 + a_2 + \ldots + a_n$ is called n^{th} partial sum of the series $\sum_{n=1}^{\infty} a_n$ and is generally denoted by S_n

i.e. $S_n = a_1 + a_2 + a_3 + \ldots + a_n$ and $<S_n>$ is called sequence of (first n) partial sums

The infinite series $\sum_{n=1}^{\infty} a_n$ is said to be convergent, divergent or oscillating according as the sequence $<S_n>$ of partial sums of the series Σa_n is convergent, divergent or oscillating

i.e. If $\lim_{n\to\infty} S_n = l$ (finite), Σa_n is cgt.

If $\lim_{n\to\infty} S_n \to +\infty$ or $-\infty$, Σa_n is dgt.

If $\lim_{n\to\infty} S_n = l$ (finite or infinite), is not unique, there Σa_n is oscillating or non-convergent.

Q 2. Define convergence, divergence and oscillation of infinite series. (PTU, Dec. 2007)

Solution. Let $<a_n>$ be a real sequence. The sum of first n terms namely $a_1 + a_2 + \ldots + a_n$ is called n^{th} partial sum of the series $\sum_{n=1}^{\infty} a_n$ and is generally denoted by S_n

i.e. $S_n = a_1 + a_2 + a_3 + \ldots + a_n$ and $<S_n>$ is called sequence of (first n) partial sums

The infinite series $\sum_{n=1}^{\infty} a_n$ is said to be convergent, divergent or oscillating according as the sequence $<S_n>$ of partial sums of the series Σa_n is convergent, divergent or oscillating

i.e. If $\lim_{n\to\infty} S_n = l$ (finite), Σa_n is cgt.

If $\lim_{n\to\infty} S_n \to +\infty$ or $-\infty$, Σa_n is dgt.

If $\lim_{n\to\infty} S_n = l$ (finite or infinite), is not unique, there Σa_n is oscillating or non-convergent.

Q 3. A series is either convergent or divergent. State true or false if false explain. (PTU, June 2007)

Solution. A series is not always cgs or dgs It can be oscillatory i.e. finitely or infinitely

e.g. $\Sigma a_n = \Sigma (-1)^n$

Here $a_n = -1 + 1 - 1 + 1 \ldots$ n terms

i.e. $a_n = \begin{cases} 0 & \text{if n is even} \\ -1 & \text{if n is odd} \end{cases}$

Q 106. Find by double integration, the area lying inside the circle $r = a \sin\theta$ and outside the cardiode $r = a(1 - \cos\theta)$. (PTU, Dec. [illegible])

Ans. Given $r = a\sin\theta \Rightarrow \sqrt{x^2+y^2} = \frac{ay}{r} \Rightarrow x^2+y^2-ay=0$

which represents a circle with centre $\left(0, \frac{a}{2}\right)$ and radius $\frac{a}{2}$

Now both circle and cardiode intersects

when $a\sin\theta = a(1-\cos\theta)$

$\Rightarrow \sin\theta + \cos\theta = 1$

on squaring both sides, we have

$\sin^2\theta + \cos^2\theta + \sin 2\theta = 1 \Rightarrow \sin 2\theta = 0$

$\Rightarrow 2\theta = 0, \pi \Rightarrow \theta = 0, \pi/2$

Thus region $R = \{(r, \theta) : a(1-\cos\theta) \le r \le a\sin\theta, 0 \le \theta \le \frac{\pi}{2}\}$

$$\text{required area} = \int_0^{\pi/2} \int_{a(1-\cos\theta)}^{a\sin\theta} r\, dr\, d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} r^2 \Big]_{a(1-\cos\theta)}^{a\sin\theta} d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} a^2\left[\sin^2\theta - (1-\cos\theta)^2\right] d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi/2}\left[\sin^2\theta - \cos^2\theta + 2\cos\theta - 1\right] d\theta$$

$$= \frac{a^2}{2}\left[\int_0^{\pi/2} -\cos 2\theta\, d\theta + \int_0^{\pi/2} 2\cos\theta - \frac{\pi}{2}\right]$$

$$= \frac{a^2}{2}\left[-\frac{\sin 2\theta}{2}\Big|_0^{\pi/2} + 2\sin\theta\Big|_0^{\pi/2} - \frac{\pi}{2}\right]$$

$$= \frac{a^2}{2}\left\{\frac{1}{2}(0-0) + 2(1-0) - \frac{\pi}{2}\right\}$$

$$= \frac{a^2}{2}\left(2 - \frac{\pi}{2}\right) = a^2\left(1 - \frac{\pi}{4}\right)$$

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Module 2

Syllabus

Sequence and series, Bolzano Weierstrass Theorem, Cauchy convergence criterion for sequence, uniform convergence, convergence of positive term series: comparison test, limit comparison test, D'Alembert's ratio test, Raabe's test, Cauchy root test, p-test, Cauchy integral test, logarithmic test, Alternating series, Leibnitz test, Power series, Taylor's series, Series for exponential, trigonometric and logarithmic functions.

BASIC CONCEPTS

Let $\{a_n\}$ be a sequence of real numbers. The expression $a_1 + a_2 + a_3 + \ldots$ is called infinite series and is denoted by $\sum_{n=1}^{\infty} a_n$ where a_n be the nth term of the series.

Partial Sum : The sum of first n terms of the series Σa_n is called sequence of partial sum of the series Σa_n and it is denoted by S_n

and $S_n = a_1 + a_2 + \ldots + a_n$

Behaviour of Series : A series Σa_n is converges or diverges or oscillates finitely or infinitely according as their sequence of partial sum S_n is cgs or dgs or oscillate finitely or infinitely

Absolutely convergent series : A series Σa_n is cgs absolutely if $\Sigma |a_n|$ converges

G.P Series : $\sum_{n=1}^{\infty} x^{n-1} = 1 + x + \ldots + x^{n-1} + \ldots$

(i) diverges if $x \ge 1$

(ii) converges if $|x| < 1$

(iii) oscillate finitely if $x = -1$

(iv) oscillate infinitely if $x < -1$

(v) cgs absolutely if $|x| < 1$

Note : If Σa_n converges then $\lim_{n\to\infty} a_n = 0$

Note : A series Σa_n is said to be +ve (positive) term series if all the terms of the series after some particular terms are +ve.

Comparison Test : Σu_n and Σv_n be two +ve term series

(i) If $\lim_{n\to\infty} \frac{u_n}{v_n} = l$ (finite and non zero) then Σu_n and Σv_n behave alike.

$$f'''(x) = -\cos x \,;\; f'''\left(\frac{\pi}{2}\right) = -1$$

$$f''''(x) = -\cos x \,;\; f''''\left(\frac{\pi}{2}\right) = 0$$

$$f^{v}(x) = \sin x \,;\; f^{v}\left(\frac{\pi}{2}\right) = 1$$

Thus by Taylor's series expansion, we have

$$f(x) = f\left(\frac{\pi}{2}\right) + \left(x - \frac{\pi}{2}\right) f'\left(\frac{\pi}{2}\right) + \frac{\left(x-\frac{\pi}{2}\right)^2}{2!} f''\left(\frac{\pi}{2}\right) + \ldots$$

$$\sin x = 1 + \left(x - \frac{\pi}{2}\right) \times 0 + \frac{\left(x-\frac{\pi}{2}\right)^2}{2!} (-1) + \ldots$$

$$= 1 - \frac{\left(x-\frac{\pi}{2}\right)^2}{2!} + \ldots$$

Q 104. Using Method of Lagrange Multipliers, find the maximum and minimum [illegible] distance of the point (3, 4, 12) from the sphere $x^2 + y^2 + z^2 = 1$. (PTU, Dec. [illegible])

Ans. Let P(x, y, z) be any point on sphere $x^2 + y^2 + z^2 = 1$

and given point be A (3, 4, 12)

Then $AP^2 = (x-3)^2 + (y-4)^2 + (z-12)^2$

To minimise or maximise AP it is convenient to minimise or maximise AP^2.

Let $f(x, y, z) = (x-3)^2 + (y-4)^2 + (z-12)^2$

subject to constraint $x^2 + y^2 + z^2 = 1$

Let $F(x, y, z) = (x-3)^2 + (y-4)^2 + (z-12)^2 + \lambda (x^2 + y^2 + z^2 - 1)$

where λ be lagrange's multiplier

for extreme points, we have

$$\frac{\partial F}{\partial x} = 0 \Rightarrow 2(x-3) + 2\lambda x = 0 \qquad \ldots(2)$$

$$\frac{\partial F}{\partial y} = 0 \Rightarrow 2(y-4) + 2\lambda y = 0 \qquad \ldots(3)$$

$$\frac{\partial F}{\partial z} = 0 \Rightarrow 2(z-12) + 2\lambda z = 0 \qquad \ldots(4)$$

from (2) $(1+\lambda)x = 3 \Rightarrow x = \frac{3}{1+\lambda}$

from eqn (3) $(1+\lambda)y = 4 \Rightarrow y = \frac{4}{1+\lambda}$

from eqn (4), $(1+\lambda)z = 12 \Rightarrow z = \frac{12}{1+\lambda}$

from (1) $\left(\frac{3}{1+\lambda}\right)^2 + \left(\frac{4}{1+\lambda}\right)^2 + \left(\frac{12}{1+\lambda}\right)^2 = 1$

$\Rightarrow \frac{169}{(1+\lambda)^2} = 1 \Rightarrow (1+\lambda)^2 = 169 \Rightarrow 1 + \lambda = \pm 13$

$\Rightarrow$

Case I: When $\lambda = \pm 13 - 1 \Rightarrow \lambda = 12, -14$

$\lambda = 12$

$$x = \frac{3}{1+12} = \frac{3}{13},\; y = \frac{4}{1+12} = \frac{4}{13},\; z = \frac{12}{1+12} = \frac{12}{13}$$

$$AP = \sqrt{\left(\frac{3}{13} - 3\right)^2 + \left(\frac{4}{13} - 4\right)^2 + \left(\frac{12}{13} - 12\right)^2}$$

$$= \sqrt{\frac{1296 + 2304 + 20736}{169}} = \sqrt{\frac{24336}{169}} = \sqrt{144} = 12$$

Case II: When $\lambda = -14$

$$x = \frac{3}{-13},\; y = \frac{4}{-13},\; z = \frac{12}{-13}$$

$$AP = \sqrt{\left(3 + \frac{3}{13}\right)^2 + \left(4 + \frac{4}{13}\right)^2 + \left(12 + \frac{12}{13}\right)^2}$$

$$= \sqrt{(9 + 16 + 144)\left(\frac{14}{13}\right)^2} = \sqrt{169 \times \frac{196}{169}} = 14$$

Thus maximum distance = 14

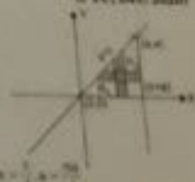
& minimum distance = 12

Q 105. Solve by changing order of integration: $\int_0^a \int_y^a \frac{x}{x^2+y^2}\,dx\,dy$, a is any positive constant. (PTU, Dec. 2020)

Ans. Here we have to integrate function w.r.t. x between the limits x = a to x = y, then integrate w.r.t. y from 0 to a. We divide the region into horizontal strips

$$R_1 = \{(x, y) : y \le x \le a,\; 0 \le y \le a\}$$

For change of order of integration we divide the region into vertical strips



$$R_2 = \{(x, y) : 0 \le y \le x,\; 0 \le x \le a\}$$

$$\int_{R_2}\int \frac{x\,dy\,dx}{x^2+y^2} = \int_0^a \int_0^x \frac{x\,dy\,dx}{x^2+y^2} = \int_0^a \frac{x}{x}\left[\tan^{-1}\frac{y}{x}\right]_0^x dx = \frac{\pi}{4}\,[\text{illegible}] = \frac{\pi a}{4}$$

Let s_n be the seq. of partial sum of Σu_n

$$s_n = u_1 + u_2 + \ldots + u_n \text{ given } \Sigma u_n \text{ cgs.}$$

$$s_n \to s \text{ as } n \to \infty$$

$$s_{n-1} \to s \text{ as } n \to \infty$$

Now $u_n = s_n - s_{n-1} \Rightarrow \lim_{n\to\infty} u_n = \lim_{n\to\infty} s_n - \lim_{n\to\infty} s_{n-1}$

$$\lim_{n\to\infty} u_n = s - s = 0$$

The converse of above theorem is not true as the series $\Sigma \frac{1}{n}$ is not cgt (∵ of p - series, p = 1)

but $\lim_{n\to\infty} \frac{1}{n} = 0$

So from above theorem a important result can be made

If $\lim_{n\to\infty} u_n \neq 0$ then the Σu_n is not cgs.

So for a +ve term series where $u_n \nrightarrow 0$ as $n \to \infty$ is dgs to + ∞.

Q 19. State Integral test for positive term series. (PTU, Dec. 2020, 2004)

Solution. It states that ∀ x ≥ 1, f(x) be monotonic decreasing function of x, non-negative

then $\sum_{n=1}^{\infty} f(n)$ or u_n and $\int_1^{\infty} f(x)\,dx$ behave alike

Now e.g. : If $f(x) = \frac{8\tan^{-1}x}{1+x^2}$

Here $f(x) \geq 0$ ∀ x ≥ 1 and f(x) be decreasing function f's

∴ Cauchy's integral test is applicable

$$\int_1^{\infty} f(x)\,dx = \int_1^{\infty} 8\tan^{-1}x \frac{1}{1+x^2}\,dx = \left[\frac{8(\tan^{-1}x)^2}{2}\right]_1^{\infty}$$

$$= 4\left[\frac{\pi^2}{4} - \frac{\pi^2}{16}\right] = \frac{3\pi^2}{4} = \text{finite}$$

Σu_n is also Converges by Cauchy's integral test.

Q 20. Test for the convergence of the series $\sum \left(\frac{n}{n+1}\right)^{n^2}$. (PTU, May 2009)

Solution. Here $u_n = \left(\frac{n}{n+1}\right)^{n^2}$

$$\lim_{n\to\infty} (u_n)^{1/n} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n = \lim_{n\to\infty} \left(\frac{1}{1+\frac{1}{n}}\right)^n = \frac{1}{e} < 1 \qquad (\because e > 1)$$

∴ by root test, the given series Σu_n converges.

[As by root test Σu_n cgs for $l < 1$ and dgs for $l > 1$

where $\lim_{n\to\infty} (u_n)^{1/n} = l$]

Q 21. Discuss the convergence of the series $x + \frac{2^2x^2}{2!} + \frac{3^3x^3}{3!} + \frac{4^4x^4}{4!} + \ldots$ (PTU, May 2006, 2007)

Solution. $u_n = \frac{n^n x^n}{n!}$; $u_{n+1} = \frac{(n+1)^{n+1} x^{n+1}}{(n+1)!}$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{n^n x^n}{n!} \cdot \frac{(n+1)!}{(n+1)^{n+1} x^{n+1}}$$

$$= \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^n} \cdot \frac{1}{x} = \frac{1}{ex}$$

cgt. for $\frac{1}{ex} > 1$, dgt. for $\frac{1}{ex} < 1$ using ratio test for $\frac{1}{ex} = 1$, Ratio test fails.

$x = \frac{1}{e}$ apply log test (∵ $\lim_{n\to\infty} \frac{u_n}{u_{n+1}}$ includes e)

$$n \log \frac{u_n}{u_{n+1}} = n\left[\log\left(\frac{1}{\left(1+\frac{1}{n}\right)^n} \cdot \frac{1}{\frac{1}{e}}\right)\right] = n\left[\log\left(1+\frac{1}{n}\right)^{-n} + \log e\right]$$

$$= -n^2 \log\left(1+\frac{1}{n}\right) + n$$

$$= -n^2\left[\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3}\right] + n$$

$$= \left(-n + \frac{1}{2} - \frac{1}{3n}\right) + n$$

$$= \frac{1}{2} - \frac{1}{3n} + \ldots$$

$\lim_{n\to\infty} n \log \frac{u_n}{u_{n+1}} = \frac{1}{2} < 1$ ∴ dgt. for $x = \frac{1}{e}$ by log test

Hence the given series Σu_n cgs for $x < \frac{1}{e}$ and dgs for $x \geq \frac{1}{e}$

$$u_n = \frac{1}{\sqrt{n^2+1}+n} \quad \text{Take } v_n = \frac{1}{n}$$

$$\underset{n\to\infty}{Lt} \frac{u_n}{v_n} = \underset{n\to\infty}{Lt} \frac{n}{\sqrt{n^2+1}+n} = \underset{n\to\infty}{Lt} \frac{1}{\sqrt{1+\frac{1}{n^2}}+1} = \frac{1}{\sqrt{1+0}+1} = \frac{1}{2} \neq 0 \text{ \& finite}$$

Also Σv_n is dgt. ($\because p = 1$) so is Σu_n (using comparison test)

Q 11. Test for convergence of the series $\sum \frac{n^2+1}{n^3+1}$. **(PTU, May [illegible])**

Solution. Compare the given series $\sum \frac{n^2+1}{n^3+1}$ with Σu_n $\therefore u_n = \frac{n^2+1}{n^3+1}$

Let $v_n = \frac{1}{n}$

$$\underset{n\to\infty}{Lt} \frac{u_n}{v_n} = \underset{n\to\infty}{Lt} \frac{n \cdot n^2\left[1+\frac{1}{n^2}\right]}{n^3\left[1+\frac{1}{n^3}\right]} = 1 \text{ [Non-zero finite real numbers]}$$

Σu_n or Σv_n behaves alike but $\Sigma v_n = \Sigma \frac{1}{n}$ is dgt ($\because p = 1$ by using p-series)

The given series is also divergent.

Q 12. What do you understand by the uniform convergence of a series? Explain with the help of one example. **(PTU, May [illegible])**

Solution. Uniform convergence : A series Σu_n converges uniformally to function $\sigma(x)$ if for a given $\epsilon > 0$ $\exists$ $m \in N$ depending upon ϵ (**independent of x**)

s.t. $|\sigma_n(x) - \sigma(x)| < \epsilon \;\; \forall n \geq m$

OR

A series $\Sigma u_n(x)$ cgs uniformally and absolutely if $\exists$ a convergent series ΣM_n of +ve constants s.t. $|u_n(x)| \leq M_n \;\; \forall n \in N$

e.g. The given series $\sum \frac{\cos nx}{n^2}$ compare with $\Sigma u_n(x)$

$$u_n(x) = \frac{\cos nx}{n^2} \Rightarrow |u_n(x)| = \left|\frac{\cos nx}{n^2}\right| \leq \frac{1}{n^2} = M_n$$

Now the series $\Sigma M_n = \Sigma \frac{1}{n^2}$ cgs ($\because p = 2 > 1$)

$\therefore$ The given series cgs uniformally and absolutely using M-test.

Q 13. If a positive term series Σu_n is convergent, then show that :

$\underset{n\to\infty}{Lt} u_n = 0$. **(PTU, May 2011, 2004 ; Dec. [illegible] ; [illegible])**

Solution. Let σ_n be the seq. of partial sum of Σu_n

$\sigma_n = u_1 + u_2 + \ldots + u_n$ given Σu_n cgs.

Now

$\sigma_n \to \sigma$ as $n \to \infty$

$\sigma_{n-1} \to \sigma$ as $n \to \infty$

$u_n = \sigma_n - \sigma_{n-1} \Rightarrow \underset{n\to\infty}{Lt} u_n = \underset{n\to\infty}{Lt} \sigma_n - \underset{n\to\infty}{Lt} \sigma_{n-1}$

$\underset{n\to\infty}{Lt} u_n = \sigma - \sigma = 0$

The converse of above theorem is not true as the series $\Sigma \frac{1}{n}$ is not cgt [illegible]

but $\underset{n\to\infty}{Lt} \frac{1}{n} = 0$

So from above theorem a important result can be made

If $\underset{n\to\infty}{Lt} u_n \neq 0$ then the Σu_n is not cgs.

So for a +ve term series where $u_n \to 0$ as $n \to \infty$ is dgs to $+\infty$

Q 17. For what values of x does the series $\sum_{n=0}^{\infty} (-1)^n (4x+1)^n$ converges absolutely **(PTU, May [illegible])**

Solution. Compare the given series $\sum_{n=0}^{\infty} (-1)^n (4x+1)^n$ with $\sum_{n=0}^{\infty} u_n$

$u_n = (-1)^n (4x+1)^n \Rightarrow |u_n| = |(4x+1)|^n$

$|u_{n+1}| = |4x+1|^{n+1}$

$$\underset{n\to\infty}{Lt} \frac{|u_{n+1}|}{|u_n|} = \underset{n\to\infty}{Lt} \frac{|4x+1|^{n+1}}{|4x+1|^n} = \underset{n\to\infty}{Lt} |4x+1|$$

The given series cgs absolutely by ratio test when $|4x+1| < 1$

$\Rightarrow -1 < 4x+1 < 1 \Rightarrow -2 < 4x < 0$

$\Rightarrow \frac{-1}{2} < x < 0$ i.e. $x \in \left(\frac{-1}{2}, 0\right)$

at $|4x+1| = 1$ The test fails i.e. at $x = 0, \frac{-1}{2}$

at $x = 0$; $u_n = (-1)^n \Rightarrow \underset{n\to\infty}{Lt} u_n \neq 0$

$\therefore \Sigma u_n$ is not cgs.

at $x = \frac{-1}{2}$; $u_n = 1 \Rightarrow \underset{n\to\infty}{Lt} u_n \neq 0$

$\therefore \Sigma u_n$ is not cgs.

Q 18. State and prove the necessary condition for the convergence of the series $\sum_{n=1}^{\infty} u_n$. **(PTU, May [illegible])**

Solution. Statement : The given series Σu_n of +ve terms is converges then $\underset{n\to\infty}{Lt} u_n = 0$

∴ Sequence of partial sum s_n of series Σu_n is neither cgt nor dgt ∴ Σu_n is finitely oscillatory as $\{s_n\}$ is bounded.

Q 4. Discuss convergence of $\sum\left(1+\frac{1}{\sqrt{n}}\right)^{-n^{3/2}}$. (PTU, May 2008)

Solution. Here $u_n = \left(1+\frac{1}{\sqrt{n}}\right)^{-n^{3/2}}$

$$u_n^{\frac{1}{n}} = \left[\left(1+\frac{1}{\sqrt{n}}\right)^{-n^{3/2}}\right]^{\frac{1}{n}} = \left(1+\frac{1}{\sqrt{n}}\right)^{\frac{-n^{3/2}}{n}} = \left(1+\frac{1}{\sqrt{n}}\right)^{-\sqrt{n}}$$

$$= \left[\left(1+\frac{1}{\sqrt{n}}\right)^{\sqrt{n}}\right]^{-1}$$

$$\lim_{n\to\infty} u_n^{\frac{1}{n}} = e^{-1} = \frac{1}{e} < 1 \qquad \left(\because \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e\right)$$

∴ Σu_n cgt using cauchy's root test

Q 5. Examine the convergence of $\sum\left(\sqrt[3]{n^3+1}-n\right)$. (PTU, May 2007)

Solution. Compare $\sum\left(\sqrt[3]{n^3+1}-n\right)$ with Σv_n

$$u_n = \sqrt[3]{n^3+1}-n = n\left[\left(1+\frac{1}{n^3}\right)^{1/3}-1\right]$$

$$= n\left[1+\frac{1}{3}\cdot\frac{1}{n^3}+\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}\left(\frac{1}{n^3}\right)^2+\text{——}-1\right]$$

$$= \frac{1}{3n^2}+\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}\cdot\frac{1}{n^5}+\text{——}$$

Take $v_n = \frac{1}{n^2}$

$$\lim_{n\to\infty}\frac{u_n}{v_n} = \frac{1}{3} \text{ (Non-zero, finite real number)}$$

Σu_n and Σv_n behave alike but $\Sigma v_n = \Sigma\frac{1}{n^2}$ is cgt series ($p = 2 > 1$) ∴ Σu_n i.e. given series converges

Q 6. State Raabe's and Logarithmic test. (PTU, May 2005)

Solution. Raabe's Test : A +ve term series Σu_n

where $\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right) = K$ cgs if $K > 1$

and dgs for $K < 1$ and test fails for $K = 1$

Logarithmic Test : A +ve term series Σu_n

where $\lim_{n\to\infty} n\log\left(\frac{u_n}{u_{n+1}}\right) = K$ cgs if $K > 1$

and dgs for $K < 1$ and test fails $K = 1$.

Q 7. State ratio test for convergence of series. (PTU, Dec. 2009 ; June 2007)

Solution. Ratio Test :

A positive term series Σu_n where $u_n > 0$

Where $\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = k$

Then Σu_n converges if $k < 1$ and dgs for $k > 1$ and $k = 1$ test fails

Q 8. Write Leibnit's rule of convergence of alternating series. (PTU, Dec. 2007)

Solution. Leibnitz's rule for alternating series : If the alternating series

$\sum(-1)^{n-1}u_n = u_1 - u_2 + u_3 - u_4 + \ldots$ ($u_n > 0$, ∀ n ∈ N) is such that

(i) $u_{n+1} \le u_n$ ∀ n and

(ii) $\lim_{n\to\infty} u_n = 0$, then the given alternating series converges

Q 9. Prove that series $\sum(-1)^{n-1}\cdot\frac{1}{n^2}$ **is absolutely convergent.** (PTU, Dec. 2008)

Solution. Compare the given series with Σu_n

$$\text{where } u_n = (-1)^{n-1}\frac{1}{n^2} \Rightarrow |u_n| = \left|(-1)^{n-1}\frac{1}{n^2}\right| = \frac{1}{n^2}$$

Now $\Sigma|u_n| = \Sigma\frac{1}{n^2}$ is convergent

because of p-series (Here $p = 2 > 1$)

∴ $\Sigma|u_n|$ is convergent ∴ Σu_n is converges absolutely

Q 10. Examine the convergence of the series $\sum\left(\sqrt{n^2+1}-n\right)$. (PTU, Dec. 2006)

Solution. $u_n = \sqrt{n^2+1}-n = \left(\sqrt{n^2+1}-n\right)\times\frac{\sqrt{n^2+1}+n}{\sqrt{n^2+1}+n} = \frac{n^2+1-n^2}{\sqrt{n^2+1}+n}$

$$= 1 + \frac{1}{n}\left(1 - \frac{4}{3}\right) + O\left(\frac{1}{n^2}\right)$$

$$= 1 - \frac{1}{3}\frac{1}{n} + O\left(\frac{1}{n^2}\right)$$

Here $\mu = -\frac{1}{3} < 1$

∴ By Gauss's test the given series Σu_n dgs at $x = \frac{1}{3}$

Hence the given series Σu_n cgs for $x < \frac{1}{3}$ and dgs for $x \geq \frac{1}{3}$.

Q 29. Test for convergence $1 + \frac{2x}{2!} + \frac{3^2 x^2}{3!} + \frac{4^3 x^3}{4!} + \frac{5^4 x^4}{5!} + \ldots\ldots$

(PTU, May 2006, 2009)

Solution. $u_n = \frac{(n+1)^n x^n}{(n+1)!}$ (leaving first term)

$$u_{n+1} = \frac{(n+2)^{n+1} x^{n+1}}{(n+2)!}$$

$$\lim_{n \to \infty} \frac{u_n}{u_{n+1}} = \lim_{n \to \infty} \frac{(n+2)!}{(n+1)!} \frac{(n+1)^n}{(n+2)^{n+1}} \frac{x^n}{x^{n+1}}$$

$$= \lim_{n \to \infty} \frac{(n+2)(n+1)!(n+1)^n}{(n+1)!(n+2)^n(n+2)} \frac{1}{x}$$

$$= \lim_{n \to \infty} \frac{n^n\left(1+\frac{1}{n}\right)^n}{n^n\left(1+\frac{2}{n}\right)^n} \frac{1}{x} = \frac{e}{e^2} \cdot \frac{1}{x} = \frac{1}{ex}$$ ∴ by ratio test

The given series cgs for $\frac{1}{ex} > 1$; dgs. for $\frac{1}{ex} < 1$

for $\frac{1}{ex} = 1$ or $x = \frac{1}{e}$ Ratio test fails

Applying log test

Now $n \log \frac{u_n}{u_{n+1}} = n \log \frac{\left(1+\frac{1}{n}\right)^n}{\left(1+\frac{2}{n}\right)^n} \cdot e$

$$= n\left[n\log\left(1+\frac{1}{n}\right) - n\log\left(1+\frac{2}{n}\right) + 1\right]$$

$$= n\left[n\left(\frac{1}{n} - \frac{1}{2n^2} + \ldots\right) - n\left(\frac{2}{n} - \frac{2}{n^2} + \ldots\right) + 1\right]$$

$$= n\left[\left(1 - \frac{1}{2n} + \ldots\right) - \left(2 - \frac{2}{n} + \ldots\right) + 1\right]$$

$$= n\left[0 + \frac{3}{2n} + \ldots\right] = \frac{3}{2} + O\left(\frac{1}{n}\right)$$

$$\lim_{n \to \infty} n \log \frac{u_n}{u_{n+1}} = \frac{3}{2} > 1$$

by logarithmic test the given series Σu_n cgs.

Hence the given series Σu_n cgs. for $x \leq \frac{1}{e}$ and dgs for $x > \frac{1}{e}$

Q 30. Verify the series $\sum \frac{4.7 \ldots\ldots (3n+1)}{1.2.3 \ldots\ldots n} x^{n-1}$ **is convergent or divergent.**

(PTU, May 2006)

Solution. $u_n = \frac{4.7 \ldots\ldots (3n+1)}{1.2.3 \ldots\ldots n} x^{n-1}$; $u_{n+1} = \frac{4.7 \ldots\ldots (3n+1)(3n+4)}{1.2 \ldots\ldots n(n+1)} x^n$

$$\lim_{n \to \infty} \frac{u_{n+1}}{u_n} = \lim_{n \to \infty} \frac{(3n+4)}{(n+1)} \frac{x^n}{x^{n-1}} = \lim_{n \to \infty} \frac{n\left(3+\frac{4}{n}\right)}{n\left(1+\frac{1}{n}\right)} x$$

by ratio test the given series cgs for $3x < 1$ i.e. $x < \frac{1}{3}$ and dgs for $3x > 1$ i.e. $x > \frac{1}{3}$ and test fails at $x = \frac{1}{3}$

We apply Gauss's Test

$$\frac{u_n}{u_{n+1}} = \frac{(n+1)3}{3n+4} = \frac{3n\left[1+\frac{1}{n}\right]}{3n\left[1+\frac{4}{3n}\right]} = \left(1+\frac{1}{n}\right)\left[1+\frac{4}{3n}\right]^{-1}$$

$$= \left(1+\frac{1}{n}\right)\left[1 - \frac{4}{3n} + O\left(\frac{1}{n^2}\right)\right]$$

$$= 1 + \frac{1}{n}\left(1 - \frac{4}{3}\right) + O\left(\frac{1}{n^2}\right)$$

$$= 1 - \frac{1}{3}\frac{1}{n} + O\left(\frac{1}{n^2}\right)$$

Sub case-1. When $p > 1 \Rightarrow p - 1 > 0$

$$\int_1^\infty f(x)\,dx = \frac{1}{1-p}\left[\frac{1}{x^{p-1}}\right]_1^\infty = \frac{1}{1-p}(0-1) = \frac{-1}{1-p} = \text{finite}$$

Sub case-2. When $p < 1 \Rightarrow 1 - p > 0$

$$\int_1^\infty f(x)\,dx = \infty$$

$\therefore \int_1^\infty f(x)\,dx$ cgs when $p > 1$ and dgs for $p < 1$

$\therefore$ The given series $\Sigma \frac{1}{n^p}$ is cgs for $p > 1$ and dgs for $p < 1$

Case-II : When $p = 1$

$$\int_1^\infty f(x)\,dx = \int_1^\infty \frac{1}{x}\,dx = \log x\Big]_1^\infty = \infty$$

$\therefore \int_1^\infty f(x)\,dx$ dgs hence the given series Σu_n dgs for $p = 1$

$\therefore$ given series $\Sigma \frac{1}{n^p}$ cgs for $p > 1$ and dgs for $p \le 1$.

Q 27. Examine the conditional convergence of the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1}$.

(PTU, Dec. 2012)

Solution. Comparing $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1}$ with $\sum_{n=1}^{\infty} (-1)^{n-1} v_n$

Here $v_n = \frac{1}{2n-1} > 0 \ \forall\ n \in N$

$$\frac{d}{dn}(v_n) = \frac{d}{dn}\left(\frac{1}{2n-1}\right) = -\frac{2}{(2n-1)^2} < 0$$

$\therefore \{v_n\}$ is monotonically decreasing sequence.

Further $\lim_{n\to\infty} v_n = \lim_{n\to\infty} \frac{1}{2n-1} = 0$

Thus by Leibnitz's test $\sum_{n=1}^{\infty} (-1)^{n-1} v_n$ i.e. $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{2n-1}$ is converges.

Let $u_n = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1} \Rightarrow |u_n| = \sum_{n=1}^{\infty} \frac{1}{2n-1} = w_n$

Let $t_n = \frac{1}{n}$

and $\lim_{n\to\infty} \frac{w_n}{t_n} = \lim_{n\to\infty} \frac{n}{2n-1} = \frac{1}{2}$ (Non-zero, finite number)

Thus by comparison test, Σw_n and Σt_n behave alike

but $\Sigma t_n = \Sigma \frac{1}{n}$ is dgs $\quad (\because \Sigma \frac{1}{n^p}$ is cgs for $p > 1$ and dgs for $p \le 1)$

Thus Σw_n is also dgs.

Hence $\Sigma |u_n|$ is not cgs.

Therefore, the given series cgs but not absolutely.

$\therefore$ The given series is conditionally cgs.

Q 28. Test for convergence $\sum \frac{4.7.\ldots\ldots(3n+1)}{1.2.\ldots\ldots n}.x^n$. **(PTU, Dec. 2008)**

Solution. Here $u_n = \frac{4.7\ldots\ldots(3n+1)}{1.2\ldots\ldots(n)} x^n$

$$u_{n+1} = \frac{4.7\ldots\ldots(3n+1)(3n+4)}{1.2\ldots\ldots n.n+1} x^{n+1}$$

$$\frac{u_{n+1}}{u_n} = \frac{3n+4}{n+1} x$$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{3n+4}{n+1} x = \lim_{n\to\infty} \frac{n\left[3+\frac{4}{n}\right]x}{n\left[1+\frac{1}{n}\right]} = 3x$$

$\therefore$ By ratio test the given series

Σu_n cgs for $3x < 1$ i.e. $x < \frac{1}{3}$ and dgs for $3x > 1$ i.e. $x > \frac{1}{3}$

at $3x = 1$ i.e. $x = \frac{1}{3}$ ratio test fails

i.e. $\frac{u_{n+1}}{u_n} = \frac{3n+4}{n+1} \cdot \frac{1}{3}$

$\Rightarrow$
$$\frac{u_n}{u_{n+1}} = \frac{3(n+1)}{3n+4} = \frac{3n\left[1+\frac{1}{n}\right]}{3n\left[1+\frac{4}{3n}\right]}$$

$$= \left[1+\frac{1}{n}\right]\left[1+\frac{4}{3n}\right]^{-1}$$

$$= \left[1+\frac{1}{n}\right]\left[1-\frac{4}{3n}+O\left(\frac{1}{n^2}\right)\right]$$

Q 22. Show that sequence converges to unique limit point if convergent.

Solution. Let $\langle a_n \rangle$ be converges to limit l and l'

∴ by def. for a given $\epsilon > 0$ however small $\exists\ m_1, m_2 \in N$

s.t. $|a_n - l| < \epsilon \ \forall n \geq m_1$

and $|a_n - l'| < \epsilon \ \forall n \geq m_2$

Let $m = \max\{m_1, m_2\}$ ∴ eq (1) gives —(1)

$|a_n - l| < \epsilon$ and $|a_n - l'| < \epsilon \ \forall n \geq m$

Now $|l - l'| = |l - a_n + a_n - l'| \leq |l - a_n| + |a_n - l'|$

$= |a_n - l| + |a_n - l'| < \epsilon + \epsilon = 2\epsilon \ \forall n \geq m$

$|l - l'| < 2\epsilon \ \forall n \geq m$

$l = l'$ hence $\langle a_n \rangle$ cgs to unique limit l.

Q 23. Test the convergence of the following series $\sum \frac{x^{n+1}}{(n+1)\sqrt{n}}$. **(PTU, Dec. 2006)**

Solution. Compare the given series with Σu_n

Here $u_n = \frac{x^{n+1}}{(n+1)\sqrt{n}}$; $u_{n+1} = \frac{x^{n+2}}{(n+2)\sqrt{n+1}}$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{x^{n+2}}{(n+2)\sqrt{n+1}} \times \frac{(n+1)\sqrt{n}}{x^{n+1}}$$

$$= \lim_{n\to\infty} x \frac{n\sqrt{n}\left(1+\frac{1}{n}\right)}{n\sqrt{n}\left(1+\frac{2}{n}\right)\sqrt{1+\frac{1}{n}}} = x$$

∴ by ratio test, the given series i.e. Σu_n is Cgs for $x < 1$ and diverges for $x > 1$

at $x = 1$, test fails

$$u_n = \frac{1}{(n+1)\sqrt{n}} \text{ choose } v_n = \frac{1}{n^{3/2}}$$

Now $\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{n^{3/2}}{(n+1)\sqrt{n}} = 1$ (Non-zero, finite number)

∴ by comparison test, Σu_n and Σv_n behave alike

but $\Sigma v_n = \Sigma \frac{1}{n^{3/2}}$ Cgs by using p-series

(Here $p = \frac{3}{2} > 1$) ∴ Σu_n also converges

Hence on combining, Σu_n Cgs for $x \leq 1$ and dgs for $x > 1$.

Q 24. Prove that series $\sum_{p=1}^{\infty} \frac{\sin px}{p^3}$ **is absolutely convergent. (PTU, Dec. 2007)**

Solution. Let us take $u_p(x) = \frac{\sin px}{p^3} \Rightarrow |u_p(x)| = \left|\frac{\sin px}{p^3}\right| \leq \frac{1}{p^3} = M_p$

Now $\Sigma M_p = \Sigma \frac{1}{p^3}$ is convergent by using p-series

∴ $\Sigma \frac{1}{p^p}$ cgs if $p > 1$ and dgs $p \leq 1$ (Here $p = 3 > 1$)

Hence by weierstrass M-test then given series $\Sigma u_p(x)$ is cgs uniformly and absolutely.

Q 25. Show that absolutely convergent series is necessarily convergent but not conversely. **(PTU, Dec. 2007)**

Solution. Let the given series Σu_n is cgs absolutely ∴ $\Sigma |u_n|$ cgs.

Hence by using Cauchy criterian of convergence for a given $\epsilon > 0$, however small $\exists\ m \in N$

s.t. $|u_{m+1}| + |u_{m+2}| + \ldots |u_{m+p}| < \epsilon \ \forall n \geq m, p \in N$

Now $|u_{m+1} + \ldots + u_{m+p}| \leq |u_{m+1}| + \ldots + |u_{m+p}| < \epsilon \ \forall n \geq m, p \in N$

∴ Σu_n is convergent by using Cauchy's criterian

but the converse is not turn

e.g. Let the series $\Sigma u_n = \Sigma (-1)^{n-1} \frac{1}{n}$ comparing with $\Sigma v_n (-1)^{n-1}$

Here $v_n = \frac{1}{n}$ and $\frac{d(v_n)}{dn} = -\frac{1}{n^2} < 0 \ \forall n \in N$

and $\lim_{n\to\infty} v_n = \lim_{n\to\infty} \frac{1}{n} = 0$

∴ both the conditions of Leibnitz's test are satisfied

Hence by Leibnitz's test, the given series cgs

but $\Sigma |u_n| = \Sigma \left|(-1)^{n-1} \frac{1}{n}\right| = \Sigma \left|\frac{1}{n}\right| = \Sigma \frac{1}{n}$

It is divergent by using p-series (Here $p = 1$)

Hence convergent series need not be absolutely convergent.

Q 26. State the integral test for convergence of series and hence discuss convergence of p-series. **(PTU, June 2007)**

Solution. Integral Test : If f(x) be non-negative, monotonic decreasing function of x $\forall x \geq 1$

s.t. $f(n) = u_n \ \forall n \in N$ then the series Σu_n and $\int_1^{\infty} f(x)\,dx$ behaves alike i.e. converges or diverges together.

Convergence of p-series (The p-series i.e. $\Sigma \frac{1}{n^p}$ converges if $p > 1$ and diverges for $0 < p \leq 1$

Let $u_n = \frac{1}{n^p} = f(n) \Rightarrow f(x) = \frac{1}{x^p}$ is monotonically decreasing and non negative $\forall x \geq 1$

∴ Integral test is applicable

Case-I : When $p \neq 1$

$$\int_1^{\infty} f(x)\,dx = \int_1^{\infty} \frac{1}{x^p}\,dx = \left.\frac{x^{1-p}}{1-p}\right|_1^{\infty} = \frac{1}{1-p}\left[x^{1-p}\right]_1^{\infty}$$

also, $\underset{n\to\infty}{Lt}\ v_n = \underset{n\to\infty}{Lt}\ \frac{1}{n} = 0$

Thus, by Leibnitz's test, given series cgs.

Q 42. State cauchy root test and use it test the convergence of the series :

$$\sum \left(\frac{n}{n+1}\right)^{n^2}.$$

(PTU, May 2010)

Solution. It states that,

If Σu_n be a +ve term series and $\underset{n\to\infty}{Lt}\ (u_n)^{1/n} = l$

Then Σu_n cgs if $l < 1$ and dgs if $l > 1$ and at $l = 1$ test fails.

Here, $u_n = \left(\frac{n}{n+1}\right)^{n^2}$ $\underset{n\to\infty}{Lt}\ (u_n)^{1/n} = \underset{n\to\infty}{Lt}\left(\frac{n}{n+1}\right)^n = \underset{n\to\infty}{Lt}\ \frac{1}{\left(1+\frac{1}{n}\right)^n}$

i.e. $\underset{n\to\infty}{Lt}\ (u_n)^{1/n} = \frac{1}{e} < 1\ (\because e > 1)$

by Cauchy's root test, the given series Σu_n cgs.

Q 43. Examine the convergence of $\frac{1}{\log 2} - \frac{1}{\log 3} + \frac{1}{\log 4} - \frac{1}{\log 5} + \ldots\ldots$

(PTU, May 2010)

Solution. The given series $= \Sigma(-1)^{n-1}\frac{1}{\log(n+1)}$, on comparing $\Sigma(-1)^{n-1}v_n$

Here, $v_n = \frac{1}{\log(n+1)} > 0\ \forall\ n \geq 1$

Now $n+2 > n+1 \Rightarrow \log(n+2) > \log(n+1)$

$\Rightarrow \frac{1}{\log(n+2)} < \frac{1}{\log(n+1)} \Rightarrow v_{n+1} < v_n$

$\therefore \{v_n\}$ is monotonically decreasing

and $\underset{n\to\infty}{Lt}\ v_n = \underset{n\to\infty}{Lt}\ \frac{1}{\log(n+1)} = 0$

By alternating series test, the given series cgs.

Q 44. Sum the series : $\sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + \ldots + \sin\left(\alpha+\overline{n-1}\beta\right)$.

(PTU, Dec. 2009)

Solution. Given, $S = \sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + \ldots + \sin\left(\alpha+\overline{n-1}\,\beta\right)$

Let, $C = \cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + \ldots + \cos\left(\alpha+\overline{n-1}\,\beta\right)$

$$C + iS = e^{i\alpha} + e^{i(\alpha+\beta)} + e^{i(\alpha+2\beta)} + \ldots + e^{i\left(\alpha+\overline{n-1}\beta\right)}$$

$$= e^{i\alpha}\left[1 + e^{i\beta} + e^{2i\beta} + \ldots + e^{i(n-1)\beta}\right]$$

$$= e^{i\alpha}\frac{\left[1-e^{in\beta}\right]}{1-e^{i\beta}} = \frac{e^{i\alpha}\left[1-\cos n\beta - i\sin n\beta\right]}{1-\cos\beta - i\sin\beta}$$

$$= e^{i\alpha}\frac{\left[2\sin^2\frac{n\beta}{2} - 2i\sin\frac{n\beta}{2}\cos\frac{n\beta}{2}\right]}{2\sin^2\frac{\beta}{2} - 2i\sin\frac{\beta}{2}\cos\frac{\beta}{2}}$$

$$= \frac{-ie^{i\alpha}\sin\frac{n\beta}{2}\left[\cos\frac{n\beta}{2} + i\sin\frac{n\beta}{2}\right]}{-i\sin\frac{\beta}{2}\left[\cos\frac{\beta}{2} + i\sin\frac{\beta}{2}\right]} = \frac{\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}\,e^{i\alpha}\,e^{i(n-1)\frac{\beta}{2}}$$

$$\therefore \quad C + iS = \frac{\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}\,e^{i\left(\alpha+\frac{\overline{n-1}}{2}\beta\right)}$$

on comparing imaginary parts on both sides ; we have

$$S = \frac{\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}\sin\left(\alpha+\frac{\overline{n-1}}{2}\beta\right)$$

Q 45. Find the interval of convergence of the series $x - \frac{x^2}{\sqrt{2}} + \frac{x^3}{\sqrt{3}} - \frac{x^4}{\sqrt{4}} + \ldots\ldots\infty$.

(PTU, Dec. 2008, 2009)

Solution. The given series can be written as $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{\sqrt{n}}$

Here, $u_n = (-1)^{n-1}\frac{x^n}{\sqrt{n}}$; $u_{n+1} = (-1)^n\frac{x^{n+1}}{\sqrt{n+1}}$

$$\underset{n\to\infty}{Lt}\left|\frac{u_{n+1}}{u_n}\right| = \underset{n\to\infty}{Lt}\left|\frac{x^{n+1}}{\sqrt{n+1}}\times\frac{\sqrt{n}}{x^n}\right| = \underset{n\to\infty}{Lt}\ |x| = |x|$$

By ratio's test, given series $\Sigma\,|u_n|$ is cgs when $|x| < 1$ i.e. $-1 < x < 1$

Thus Σu_n is cgs absolutely i.e. Σu_n is cgs when $-1 < x < 1$

When $|x| = 1$ i.e. $x = \pm 1$, test fails.

Let us take $v_n = \frac{1}{n^2}$

$$\lim_{n \to \infty} \frac{u_n}{v_n} = \lim_{n \to \infty} \frac{2n^2}{n^2\left[\sqrt{1+\frac{1}{n^4}} + \sqrt{1-\frac{1}{n^4}}\right]} = 1 \quad \text{(which is non-zero and finite)}$$

Therefore, by comparison test, both Σu_n and Σv_n behave alike.

But $\Sigma v_n = \sum \frac{1}{n^2}$ is convergent by using p-series (here $p = 2 > 1$)

Therefore, Σu_n is also convergent.

Q 38. Discuss the convergence of the series $\sum \frac{2^n - 2}{2^n + 1} x^{n-1}$ $(x > 0)$. (PTU, Dec. 2009)

Solution. Comparing the given series with Σu_n

Here $u_n = \frac{2^n - 2}{2^n + 1} x^{n-1}$ $\quad (\because x > 0)$

$$u_{n+1} = \frac{2^{n+1} - 2}{2^{n+1} + 1} x^n$$

Now
$$\lim_{n \to \infty} \frac{u_{n+1}}{u_n} = \lim_{n \to \infty} \frac{2^{n+1} - 2}{2^{n+1} + 1} \times \frac{2^n + 1}{2^n - 2} x$$

$$= \lim_{n \to \infty} \frac{1 - \frac{2}{2^{n+1}}}{1 + \frac{1}{2^{n+1}}} \cdot \lim_{n \to \infty} \frac{\left(1 + \frac{1}{2^n}\right)}{1 + \frac{1}{2^{n-1}}} x$$

$$= \frac{1 - 0}{1 + 0} \times \frac{1 + 0}{1 - 0} x \qquad \left[\because \lim_{n \to \infty} \frac{1}{2^n} = 0\right]$$

By Ratio test, the given series cgt for $x < 1$ and dgs for $x > 1$ while at $x = 1$, test fails.

When $x = 1$, $u_n = \frac{2^n - 2}{2^n + 1} \Rightarrow \lim_{n \to \infty} u_n = \lim_{n \to \infty} \frac{1 - \frac{1}{2^{n-1}}}{1 + \frac{1}{2^n}} = 1 \neq 0$

also Σu_n is a positive term series with $\lim_{n \to \infty} u_n \neq 0$.

It dgs to ∞.

Hence the given series Σu_n cgs for $x < 1$ and dgs for $x \geq 1$.

Q 39. Test the convergence of the series $\frac{1}{1.2.3} + \frac{3}{2.3.4} + \frac{5}{3.4.5} + \ldots\ldots$

(PTU, Dec. 2010 ; May 2009)

Solution. Comparing the given series with Σu_n :

where $u_n = \frac{n\text{th term of } 1, 3, 5 \ldots}{(n\text{th term of } 1, 2, 3 \ldots)\,(n\text{th term of } 2, 3, 4 \ldots)\,(n\text{th term of } 3, 4, 5 \ldots)}$

$$u_n = \frac{2n - 1}{n(n+1)(n+2)}$$

Let $v_n = \frac{1}{n^2}$

$$\lim_{n \to \infty} \frac{u_n}{v_n} = \lim_{n \to \infty} \frac{n^2 \cdot n\left(2 - \frac{1}{n}\right)}{n \cdot n\left(1 + \frac{1}{n}\right) n\left(1 + \frac{2}{n}\right)} = 2 \text{ (Non-zero, finite number)}$$

using comparison test, both series Σu_n and Σv_n behave alike

but $\Sigma v_n = \Sigma \frac{1}{n^2}$ is cgs by p-series $\qquad \left(\because \Sigma \frac{1}{n^p} \text{ cgs if } p > 1 \text{ and dgs if } p \leq 1\right)$

The given series Σu_n is also cgs. $\qquad$ (here $p = 2$)

Q 40. Show that the series $\sum_{n=1}^{\infty} \frac{\sin\left(x^2 + nx\right)}{n(n+2)}$ for all real x, is uniformly convergent.

(PTU, May 2009)

Solution. Here $u_n(x) = \frac{\sin\left(x^2 + nx\right)}{n(n+2)}$

Now
$$|u_n(x)| = \left|\frac{\sin\left(x^2 + nx\right)}{n(n+2)}\right| \leq \frac{1}{n(n+2)} < \frac{1}{n^2} = M_n \qquad (\because |\sin\theta| \leq 1 \ \forall \theta)$$

Now $\Sigma M_n = \Sigma \frac{1}{n^2}$ cgs $\qquad$ ($\because$ of p - series, here $p = 2 > 1$)

Hence by M – test the given series cgs. uniformally $\forall x \in R$

Q 41. Examine the convergence of $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots\ldots$ (PTU, Dec. 2009)

Solution. The given series can be written as $\Sigma (-1)^{n-1} \frac{1}{n}$

It is a alternating series, on comparing with $\Sigma (-1)^{n-1} v_n$

Here $v_n = \frac{1}{n}$ and $n + 1 > n \Rightarrow \frac{1}{n+1} < \frac{1}{n} \Rightarrow v_{n+1} - v_n < 0$

$\{v_n\}$ is monotonically decreasing sequence

Here $\mu = \frac{-1}{3} < 1$

$\therefore \Sigma u_n$ dgs by using Gauss's test.

$\therefore$ The given series cgs for $x < \frac{1}{3}$ and dgs for $x \geq \frac{1}{3}$

Q 31. Test the following series for uniform convergence, $\sum \frac{(\cos nx)}{n^3}$ for $-\pi \leq x \leq \pi$. (PTU, Dec. 2009)

Solution. Here $u_n(x) = \frac{\cos nx}{n^3}$

$$\Rightarrow \quad |u_n(x)| = \left|\frac{\cos nx}{n^3}\right| \leq \frac{1}{n^3} = M_n \quad [\because |\cos nx| \leq 1]$$

Now $\Sigma M_n = \Sigma \frac{1}{n^3}$. It is a convergent series $\because$ of p-series (here $p = 3 > 1$)

By Weierstrass M-test $\therefore$ The given series i.e. $\Sigma u_n(x) = \sum \frac{\cos nx}{n^3}$

is also uniformally converges $\forall x$.

Q 32. Discuss the covergence of the series :

$$1 + \frac{2^2}{3^2} + \frac{2^2.4^2}{3^2.5^2} + \frac{2^2.4^2.6^2}{3^2.5^2.7^2} + \ldots\ldots \infty.$$ (PTU, May 2004)

Solution. $$u_n = \frac{2^2.4^2.6^2 \ldots (2n)^2}{3^2.5^2.7^2 \ldots (2n+1)^2}$$ (leaving 1st term)

$$u_{n+1} = \frac{2^2.4^2.6^2 \ldots (2n)^2 (2n+2)^2}{3^2.5^2.7^2 \ldots (2n+1)^2 (2n+3)^2}$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \left(\frac{2n+3}{2n+2}\right)^2 = \lim_{n\to\infty} \left(\frac{2+\frac{3}{n}}{2+\frac{2}{n}}\right)^2 = \left(\frac{2}{2}\right)^2 = 1$$

Ratio test fails.

Applying Raabe's Test.

$$n\left[\frac{u_n}{u_{n+1}} - 1\right] = n\left[\left(\frac{2n+3}{2n+2}\right)^2 - 1\right] = n\left[\frac{5+4n}{(2n+2)^2}\right] = \frac{\frac{5}{n}+4}{\left(2+\frac{2}{n}\right)^2}$$

$$\lim_{n\to\infty} n\left[\frac{u_n}{u_{n+1}} - 1\right] = \frac{4}{2^2} = 1$$

Raabe's test fails.

Applying Logarithmic Test

$$n \log \frac{u_n}{u_{n+1}} = n\,(2 \log (2n+3) - 2 \log (2n+2))$$

$$= 2n\left[\log\left(1+\frac{3}{2n}\right) - \log\left(1+\frac{1}{n}\right)\right]$$

$$= 2n\left[\left(\frac{3}{2n} - \frac{\left(\frac{3}{2n}\right)^2}{2} + \ldots\right) - \left(\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \ldots\right)\right]$$

$$= 2\left[\left(\frac{3}{2} - 1\right) + \left(\frac{-9}{8n} + \frac{1}{2n}\right) \ldots\right]$$

$$\lim_{n\to\infty} n \log \frac{u_n}{u_{n+1}} = 2\left(\frac{1}{2}\right) = 1$$

Logarithmic test fails.

Applying Gauss's Test

$$\frac{u_n}{u_{n+1}} = \frac{(2n+3)^2}{(2n+2)^2} = \frac{\left(1+\frac{3}{2n}\right)^2}{\left(1+\frac{1}{n}\right)^2}$$ (Note : Always express in terms of $\frac{1}{n}$)

$$= \left(1 + \frac{9}{4n^2} + \frac{3}{n}\right)\left(1+\frac{1}{n}\right)^{-2} = \left(1 + \frac{3}{n} + \frac{9}{4n^2}\right)\left(1 - \frac{2}{n} + \ldots\right)$$

$$= 1 - \frac{2}{n} + \frac{3}{n} - \frac{6}{n^2} + \ldots$$

$$= 1 + \frac{1}{n} + O\left(\frac{1}{n^2}\right)$$

here $\mu = 1$ $\therefore$ The given series dgs by gauss's test.

Q 33. Test convergence/diverge of the series $\sum_{n=1}^{\infty} \left[\sqrt{(n^4+1)} - \sqrt{(n^4-1)}\right]$. (PTU, Dec. 2013)

Solution. Compare the given series with Σv_n

$$u_n = \left(\sqrt{n^4+1} - \sqrt{n^4-1}\right) \frac{\left(\sqrt{n^4+1} + \sqrt{n^4-1}\right)}{\left(\sqrt{n^4+1} + \sqrt{n^4-1}\right)}$$

$$= \frac{n^4+1-(n^4-1)}{\sqrt{n^4+1} + \sqrt{n^4-1}} = \frac{2}{\sqrt{n^4+1} + \sqrt{n^4-1}}$$

Q 51. Test for convergence the series :

$$\frac{\alpha}{\beta} + \frac{1+\alpha}{1+\beta} + \frac{(1+\alpha)(2+\alpha)}{(1+\beta)(2+\beta)} + \ldots$$

(PTU, May 2011)

Solution. Neglecting first term,

$$u_n = \frac{(1+\alpha)(2+\alpha)\ldots(n+\alpha)}{(1+\beta)(2+\beta)\ldots(n+\beta)}$$

$$u_{n+1} = \frac{(1+\alpha)(2+\alpha)\ldots(n+1+\alpha)}{(1+\beta)(2+\beta)\ldots(n+1+\beta)}$$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{n+1+\alpha}{n+1+\beta} = 1$$

Ratio test fails, we apply Gauss's test

$$\frac{u_n}{u_{n+1}} = \frac{n+1+\beta}{n+1+\alpha} = \frac{\left(1+\frac{1+\beta}{n}\right)}{\left(1+\frac{1+\alpha}{n}\right)} = \left(1+\frac{1+\beta}{n}\right)\left(1+\frac{1+\alpha}{n}\right)^{-1}$$

$$= \left[1+\frac{1+\beta}{n}\right]\left[1-\frac{1+\alpha}{n}\right]$$

$$= \left[1+\frac{1}{n}(1+\beta-1-\alpha)+O\left(\frac{1}{n^2}\right)\right] = 1+\frac{(\beta-\alpha)}{n}+O\left(\frac{1}{n^2}\right)$$

By Gauss test, the given series cgs for $\mu = \beta-\alpha > 1$ and cgs for $\mu = \beta - \alpha \le 1$.

Q 52. Test the convergence of the series $\sum_{n=1}^{\infty} \frac{\cos n\pi}{n^2+1}$. (PTU, May 2012)

Solution. The given series becomes $\sum \frac{(-1)^n}{n^2+1}$ as $\cos n\pi = (-1)^n$

On comparing with $\Sigma(-1)^{n-1}(-v_n)$

$$v_n = \frac{1}{n^2+1} > 0$$

Now $$\frac{dv_n}{dn} = \frac{-2n}{(n^2+1)^2} < 0 \ \forall\, n \ge 1$$

Therefore, $\{v_n\}$ is monotonically decreasing sequence

Now $$\lim_{n\to\infty} v_n = \lim_{n\to\infty} \frac{1}{n^2+1} = 0$$

Therefore, by alternating series test the given series converges.

Q 53. Check the convergence of the following sequences whose nth term is given by $a_n = \left(\frac{3n+1}{3n-1}\right)^n$. (PTU, Dec. 2020)

Ans. Given $$a_n = \left(\frac{3n+1}{3n-1}\right)^n$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left[1+\frac{2}{3n-1}\right]^n = \lim_{n\to\infty}\left[1+\frac{2}{3n-1}\right]^{\frac{3n-1}{2}\cdot\frac{2n}{3n-1}}$$

$$= \left[\lim_{n\to\infty}\left[1+\frac{2}{3n-1}\right]^{\frac{3n-1}{2}}\right]^{\frac{2n}{3n-1}} \qquad \left[\because \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e\right]$$

$$= e^{\lim_{n\to\infty}\frac{2n}{3n-1}} = e^{2/3}, \text{ which is finite}$$

Then the given sequence convergences.

Q 54. Check the convergence of the series $\sum_{n=1}^{\infty} \frac{\sqrt{n+1}-\sqrt{n}}{n^{3/2}}$. (PTU, Dec. 2020)

Ans. Here $$u_n = \frac{\sqrt{n+1}-\sqrt{n}}{n^{3/2}} = \frac{\sqrt{n+1}-\sqrt{n}}{n^{3/2}} \times \frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}$$

$$= \frac{n+1-n}{n^{3/2}(\sqrt{n+1}+\sqrt{n})} = \frac{1}{n^{3/2}(\sqrt{n+1}+\sqrt{n})}$$

Take $$v_n = \frac{1}{n^2}$$

$$\lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\frac{n^2}{n^{3/2}(\sqrt{n+1}+\sqrt{n})} = \lim_{n\to\infty}\frac{n^2}{n^{3/2}\sqrt{n}\left(\sqrt{1+\frac{1}{n}}+1\right)}$$

$$= \lim_{n\to\infty}\frac{1}{\left(\sqrt{1+\frac{1}{n}}+1\right)} = \frac{1}{2} \text{ (finite, non zero)}$$

Then by comparison Test, Σu_n and Σv_n behave alike

$$\text{but } \Sigma v_n = \Sigma\frac{1}{n^2} \text{ cgs As } \Sigma\frac{1}{n^p} \text{ cgs for } p > 1$$

and dgs for $p \le 1$, Here $p = 2 > 1$

Thus given series Σu_n converges.

□□□

∴ radius of convergence $= r = \frac{1}{\rho} = \frac{1}{3} \neq 0, \infty$

Thus interval of convergence $(x_0 - r, x_0 + r)$

i.e. $\left(-\frac{1}{3}-\frac{1}{3}, -\frac{1}{3}+\frac{1}{3}\right)$ i.e. $\left(-\frac{2}{3}, 0\right)$

Also, $\underset{n \to \infty}{Lt} \left|\frac{u_{n+1}}{u_n}\right| = \underset{n \to \infty}{Lt} \left|\frac{(3x+1)^{n+2}}{2n+4} \times \frac{2n+2}{(3x+1)^{n+1}}\right| = |3x+1|$

∴ By ratio test $\Sigma |u_n|$ cgs if $|3x+1| < 1$

i.e. $-1 < 3x+1 < 1 \Rightarrow -\frac{2}{3} < x < 0$ i.e. $x \in \left(-\frac{2}{3}, 0\right)$

∴ Σu_n absolutely cgs if $x \in \left(-\frac{2}{3}, 0\right)$

and test fails if $|3x+1| = 1$ i.e. $x = 0, -\frac{2}{3}$

When $x = 0$, Given series becomes $\Sigma \frac{1}{2n+2}$

i.e. $u_n = \frac{1}{2n+2}$; we take $v_n = \frac{1}{n}$

s.t. $\underset{n \to \infty}{Lt} \frac{u_n}{v_n} = \underset{n \to \infty}{Lt} \frac{n}{2n+2} = \frac{1}{2}$ (non-zero, finite)

∴ Σu_n and Σv_n behave alike, by comparison test

but $\Sigma v_n = \Sigma \frac{1}{n}$ is dgs by using p-series (∵ $p = 1$)

∴ Σu_n is also dgs.

When $x = -\frac{2}{3}$; Given series becomes, $\sum \frac{(-1)^{n-1}}{2n+2}$

Here, $v_n = \frac{1}{2n+2}$ on comparing given series with $\Sigma(-1)^{n-1} v_n$

and $\underset{n \to \infty}{Lt} v_n = \underset{n \to \infty}{Lt} \frac{1}{2n+2} = 0$

also $2n+4 > 2n+2 \Rightarrow \frac{1}{2n+4} < \frac{1}{2n+2} \Rightarrow v_{n+1} - v_n < 0$

∴ $\{v_n\}$ is monotonically decreasing.

Thus by alternating series test, the given series cgs.

Also every absolutely cgt series is convergent

∴ Given series cgs for $x \in \left[-\frac{2}{3}, 0\right]$.

Therefore at $x = -\frac{2}{3}$, series is cgs but not absolutely.

∴ The given series conditionally cgs at $x = -\frac{2}{3}$

Q 49. What is Alternating Series? Explain the method to test the convergence of an alternating series. (PTU, Dec. 2018)

Solution. A series which contains alternative positive and negative signs is called alternating series.

If the alternating series $\Sigma(-1)^{n-1} u_n = u_1 - u_2 + u_3 - u_4 + \ldots (u_n > 0, \forall n \in N)$ is such that

(i) $u_{n+1} \le u_n \ \forall n$

(ii) $\lim\limits_{n \to \infty} u_n = 0$, then the series converges.

Cor. If $u_n \to u \neq 0$ then the alternating series $\Sigma(-1)^{n-1} u_n = u_1 - u_2 + u_3 - u_4 \ldots$ oscillates finitely.

Q 50. State, with reasons, the values of x for which the series

$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5}$ **— converges.** (PTU, Dec. 2018)

Solution. The given series can be written as

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = \sum_{n=1}^{\infty} u_n$$

i.e.
$$|u_n| = \frac{|(-1)^{n-1} x^n|}{n} = \frac{|x^n|}{n}$$

$$\frac{|u_{n+1}|}{|u_n|} = \frac{|x|^{n+1}}{n+1} \times \frac{n}{|x|^n} = \frac{n}{n+1}|x|$$

$$\underset{n \to \infty}{Lt} \frac{|u_{n+1}|}{|u_n|} = |x|$$

Therefore, by ratio test the series $\Sigma |u_n|$ converges for $|x| < 1$ i.e. $-1 < x < 1$ and diverges for $|x| > 1$

at $|x| = 1$ ratio test fails.

For $|x| = 1$ i.e., $x = \pm 1$

When $x = 1$, the series $\Sigma(-1)^{n-1} \frac{1}{n}$ which is convergent by Leibnitz's test

When $x = -1$, the series $\Sigma(-1)^{n-1} \frac{(-1)^n}{n} = -\Sigma \frac{1}{n}$

The given series becomes divergent (∵ of p-series here $p = 1$)

Therefore, the given series becomes convergent for $-1 < x \le 1$.

When $x = 1$, $\Sigma u_n = \Sigma(-1)^{n-1}\frac{1}{\sqrt{n}}$, Here $v_n = \frac{1}{\sqrt{n}} \Rightarrow \lim_{n\to\infty} v_n = 0$

Now, $v_{n+1} = \frac{1}{\sqrt{n+1}}$ ∴ $v_{n+1} - v_n < 0$ Thus $\{v_n\}$ is monotonically decreasing

Thus, alternating series test given series $\Sigma(-1)^{n-1}v_n$ cgs

When $x = -1$, $\Sigma u_n = -\Sigma\frac{1}{\sqrt{n}}$ which is divergent ∵ of p-series (Here $p = \frac{1}{2} < 1$)

∴ Given series Σu_n cgs for $-1 < x \leq 1$

Q 46. Test the convergence of the series :

(i) $\sqrt{n^2+1} - n$ (ii) $\frac{1}{1.3} + \frac{2}{3.5} + \frac{3}{5.7} + \ldots\ldots\infty$ (PTU, Dec. 2009)

Solution.

(i) $$u_n = \sqrt{n^2+1} - n = \left(\sqrt{n^2+1} - n\right) \times \frac{\sqrt{n^2+1}+n}{\sqrt{n^2+1}+n} = \frac{n^2+1-n^2}{\sqrt{n^2+1}+n}$$

$$u_n = \frac{1}{\sqrt{n^2+1}+n} \quad v_n = \frac{1}{n}$$

$$\lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\frac{n}{\sqrt{n^2+1}+n} = \lim_{n\to\infty}\frac{1}{\sqrt{1+\frac{1}{n^2}}+1} = \frac{1}{\sqrt{1+0}+1} = \frac{1}{2} \neq 0 \text{ \& finite}$$

Also Σv_n is dgt. ∵ $p = 1$) so is Σu_n. (using comparison test)

(ii) Here, $u_n = \frac{n}{(2n-1)(2n+1)}$, Let $v_n = \frac{n}{n^2} = \frac{1}{n}$

$$\lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\frac{n^2}{(2n-1)(2n+1)} = \lim_{n\to\infty}\frac{1}{\left(2-\frac{1}{n}\right)\left(2+\frac{1}{n}\right)}$$

$$= \frac{1}{(2-0)(2+0)} = \frac{1}{4} \neq 0 \text{ \& finite}$$

∴ By comparison test, Σu_n & Σv_n cgs. & dgs. together.

Now Σv_n is dgt. ∵ $p = 1$

∴ Σu_n is also dgs.

Q 47. Discuss the convergence/divergence of the series

(i) $\sum_{n=1}^{\infty}\frac{(\ln n)^2}{n^{3/2}}$ (ii) $\sum_{n=1}^{\infty}\frac{(n!)}{(n^n)^2}$ (PTU, May 2010)

Solution. (i) Now, $\frac{(\ln n)^2}{n^{1/4}} \to 0$ as $n \to \infty$ $\left[∵ \lim_{n\to\infty}\frac{(\ln n)^p}{n^q} = 0 \text{ if } q > 0\right]$

∴ by def. ∃ $m \in N$ s.t. $\frac{(\ln n)^2}{n^{1/4}} < 1$

$\Rightarrow \frac{(\ln n)^2}{n^{3/2}} < \frac{n^{1/4}}{n^{3/2}} \Rightarrow \frac{(\ln n)^2}{n^{3/2}} < \frac{1}{n^{5/4}}$

$\Rightarrow \Sigma u_n < \Sigma\frac{1}{n^{5/4}}$, but $\Sigma\frac{1}{n^{5/4}}$ is cgs ∵ of p-series (here $p = 5/4 > 1$)

∴ by comparison test, Σu_n is also cgs.

(ii) Here $u_n = \frac{n!}{(n^n)^2}$ $u_{n+1} = \frac{(n+1)!}{(n+1)^{2(n+1)}}$

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = \lim_{n\to\infty}\frac{(n+1)!}{(n+1)^{2n+2}} \times \frac{n^{2n}}{n!} = \lim_{n\to\infty}\frac{n^{2n}}{(n+1)^{2n+1}}$$

$$= \lim_{n\to\infty}\left[\frac{n}{n+1}\right]^{2n}\frac{1}{n+1}$$

$$= \lim_{n\to\infty}\frac{1}{\left[\left(1+\frac{1}{n}\right)^n\right]^2}\frac{1}{n+1} = 0 < 1$$

∴ by ratio test, the given series Σu_n converges

Q 48. Find the radius and interval of convergence of the series.

$$\sum\frac{(3x+1)^{n+1}}{2n+2}$$

Further, for what values of x (if any) does the series converges (i) absolutely (ii) conditionally. (PTU, May 2010)

Solution. The given series $\sum\frac{(3x+1)^{n+1}}{2n+2}$ can be written as

$\sum\frac{3^{n+1}}{2n+2}\left(x+\frac{1}{3}\right)^{n+1}$ compare with $\Sigma a_n(x-x_0)^{n+1}$

Here, $x_0 = -\frac{1}{3}$ and $a_n = \frac{3^{n+1}}{2n+2}$

$$\mu = \lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{3^{n+2}}{2n+4} \times \frac{2n+2}{3^{n+1}} = 3$$

QUESTION-ANSWERS

Q 1. Solve : $\sec^2 x \tan y \, dx + \sec^2 y \tan x \, dy = 0$. (PTU, Dec. [illegible])

Ans. Separate the variables, we get

$$\frac{\sec^2 x}{\tan x} dx + \frac{\sec^2 y}{\tan y} dy = 0; \text{ on integrating, we get}$$

$\log \tan x + \log \tan y = \log c \Rightarrow \tan x \tan y = c$ be the req. sol.

Q 2. Solve : $x \cos x \cos y + \sin y \dfrac{dy}{dx} = 0$. (PTU, Dec. [illegible])

Ans. $x \cos x \cos y + \sin y \dfrac{dy}{dx} = 0 \Rightarrow \dfrac{\sin y}{\cos y} dy + x \cos x \, dx = 0$

$\int \dfrac{\sin y}{\cos y} dy + \int x \cos x \, dx = 0$; on integrating, we get

$\Rightarrow -\log |\cos y| + x \sin x + \cos x = c$ is req. solution.

Q 3. Explain briefly how to solve the differential equation :

$$\frac{dy}{dx} = \frac{ax + by + c}{a_1 x + b_1 y + c_1} \text{ when } \frac{a}{a_1} \neq \frac{b}{b_1}$$ (PTU, Dec. [illegible])

Ans. Put $x = X + h \Rightarrow dx = dX$

$y = Y + k \Rightarrow dy = dY$

The given eq. becomes $\dfrac{dY}{dX} = \dfrac{a(X+h) + b(Y+k) + c}{a_1(X+h) + b_1(Y+k) + c_1}$

$$\Rightarrow \frac{dY}{dX} = \frac{aX + bY + ah + bk + c}{a_1 X + b_1 Y + a_1 h + b_1 k + c_1} \quad \text{—(1)}$$

Choose h, k so that eq (1) is homogeneous i.e. $ah + bk + c = 0$ —(2)

and $a_1 h + b_1 k + c_1 = 0$ —(3)

on solving (2) and (3), we get $\dfrac{h}{bc_1 - b_1 c} = \dfrac{k}{ac_1 - a_1 c} = \dfrac{1}{ab_1 - a_1 b}$

i.e. $h = \dfrac{bc_1 - b_1 c}{ab_1 - a_1 b}$ and $k = \dfrac{ac_1 - a_1 c}{ab_1 - a_1 b}$ Now $\dfrac{a}{a_1} \neq \dfrac{b}{b_1}$ i.e. $ab_1 - a_1 b \neq 0$

so h and k are finite

$\dfrac{dY}{dX} = \dfrac{aX + bY}{a_1 X + b_1 Y}$ then put $Y = vX$ and apply method of homogeneous diff. eq.

Q 4. Solve $(3y + 2x + 4) dx - (4x + 6y + 5) dy = 0$. (PTU, Dec. [illegible])

Ans. Given diff. eq. be $\dfrac{dy}{dx} = \dfrac{(2x + 3y + 4)}{4x + 6y + 5}$

put $2x + 3y = t \Rightarrow 2 + 3\dfrac{dy}{dx} = \dfrac{dt}{dx}$

$\Rightarrow \quad \dfrac{1}{3}\left(\dfrac{dt}{dx} - 2\right) = \dfrac{t + 4}{2t + 5}$

$\Rightarrow \quad \dfrac{dt}{dx} = \dfrac{3t + 12}{2t + 5} + 2 \Rightarrow \dfrac{dt}{dx} = \dfrac{7t + 22}{2t + 5}$

$\Rightarrow \quad \dfrac{(2t + 5)}{7t + 22} dt = dx$, on integrating we get

$$\int \frac{2(t + 5/2)}{7t + 22} dt = x + c \Rightarrow \int \frac{2}{7} \frac{(7t + 35/2)}{7t + 22} dt = x + c$$

$\Rightarrow \quad \dfrac{2}{7}\int \left[1 - \dfrac{9/2}{7t + 22}\right] dt = x + c \Rightarrow \dfrac{2}{7}\left[t - \dfrac{9}{14} \log (7t + 22)\right] = x + c$

$\Rightarrow \quad \dfrac{2}{7}\left[2x + 3y - \dfrac{9}{14} \log (14x + 21y + 22)\right] = x + c$

Q 5. Solve the following differential equations.

(i) $x \dfrac{dy}{dx} = y + \sqrt{x^2 + y^2}$

(ii) $y = xy' + (y')^2$ (PTU, May [illegible])

Ans. (i) $\dfrac{dy}{dx} = \dfrac{y}{x} + \sqrt{1 + \dfrac{y^2}{x^2}}$; put $y = vx \Rightarrow \dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

$$v + x\frac{dv}{dx} = v + \sqrt{1 + v^2} \Rightarrow \frac{dv}{\sqrt{1 + v^2}} = \frac{dx}{x}$$

On integrating we get

$$\log \left|v + \sqrt{1 + v^2}\right| = \log x + \log c$$

$\Rightarrow \quad \log \left|\dfrac{v + \sqrt{1 + v^2}}{x}\right| = \log c \Rightarrow \log \left|\dfrac{y + \sqrt{x^2 + y^2}}{x^2}\right| = \log c$

(iii) A diff. eq. of the form $\frac{dy}{dx} + Py = Q$ where P, Q are functions of x alone.

Here integrating factor = I.F. = $e^{\int P dx}$

and solution is given by $y \cdot e^{\int P dx} = \int Q e^{\int P dx} dx + C$

or We can make linear diff. eq. in x i.e. $\frac{dx}{dy} + Px = Q$

Where P and Q are functions of y alone

Here I.F. = $e^{\int P dy}$ and solution is given by

$$x e^{\int P dy} = \int Q e^{\int P dy} dy + C$$

Bernoulli's form : A diff. eq. is of the form $\frac{dy}{dx} + Py = Qy^n$

Where P, Q are functions of x alone.

Here divide throughout by y^n then put $y^{1-n} = z$ we get linear in z.

Exact differential equation : A diff. equation of the form Mdx + Ndy = 0 where M, N are functions of x, y is said to be exact if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ and its solution is given by

$$\int M dx + \int (\text{term containing } y \text{ alone})\, dy = C$$

Equations reducible to exact eq : If the diff. eq. is not exact then we multiply the whole eq. by I.F.

INTEGRATING FACTOR

When the diff. eq. is not exact we multiply that eq. by a factor so that it becomes exact that factor is called integrating factor.

We can find I.F by Inspection.

Terms	I.F	Exact differential
x dx + y dy	1	d (xy)
x dy − y dx	(i) $\frac{1}{x^2}$	$d\left(\frac{y}{x}\right)$
	(ii) $\frac{1}{y^2}$	$d\left(-\frac{x}{y}\right)$
	(iii) $\frac{1}{xy}$	$d\left(\log\left(\frac{y}{x}\right)\right)$

	(iv) $\frac{1}{x^2+y^2}$	$d\left(\tan^{-1}\frac{y}{x}\right)$
x dy + y dx	$\frac{1}{(xy)^n}$	$d\left(\frac{-1}{(n-1)(xy)^{n-1}}\right), n \neq 1$
x dx + y dy	$\frac{1}{(x^2+y^2)^n}$	$d\left(\frac{-1}{2(n-1)(x^2+y^2)^{n-1}}\right), n \neq 1$

Five Rules for Finding Integrating factor and hence reducing the eqn. to exact eqn.

Rule I : If the eq. M dx + N dy = 0 is homogeneous eq. in x and y. Then $\frac{1}{Mx + Ny}$ be the I.F. provided $Mx + Ny \neq 0$

Rule II : If the eq. M dx + N dy = 0 is of the form $f(xy)\, y\, dx + g(xy)\, x\, dy = 0$. Then $\frac{1}{Mx - Ny}$ be the I.F. provided $Mx - Ny \neq 0$.

Rule III : If $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M} = f(y)$ (function of y alone)

Then I.F. = $e^{\int g(y) dy}$

Rule IV : $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = f(x)$ (i.e. a function of x alone)

Then I.F. = $e^{\int f(x) dx}$

Ryle V : If the eq. M dx + N dy = 0 is the form

$$x^a y^b (m\, y dx + n\, x dy) + x^{a'} y^{b'} (m' y\, dx + n' x\, dy) = 0$$

Then I.F. = $x^h y^k$ where $\frac{a+h+1}{m} = \frac{b+k+1}{n}$

and $\frac{a'+h+1}{m'} = \frac{b'+k+1}{n'}$

FOR NOTES

Module 3

Syllabus

Exact, linear and Bernoulli's equations, Euler's equations, Equations not of first degree: equations solvable for p, equations solvable for y, equations solvable for x and Clairaut's type.

BASIC CONCEPTS

Differential Equation : It is an equation which contains differential coefficients or differentials.

e.g. $\frac{dy}{dx} = x\frac{d^2y}{dx^2} + 1,\ \frac{dy}{dx} + Py = 2$ etc.

Solution of first order and first degree eq : It can be solved by following methods (i) Variable separable (ii) Homogeneous diff. eq. (iii) Linear diff. eq.

(i) If in an diff. eq. it is possible to collect all functions of x and dx on one side and all functions by y and dy on other side Then diff eq. is of the form $f(y)\,dy = g(x)\,dx$ on integrating we get

$$\int f(y)\,dy = \int g(x)\,dx + c \text{ as its solution.}$$

(ii) A diff. eq. is of the form $\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)}$ where $f(x, y)$ and $g(x, y)$ are functions of same degree.

Here we put $y = vx,\ \frac{dy}{dx} = v + x\frac{dv}{dx}$

and the separate the variable v and x and integrate.

Equations reducible to Homogeneous form

A diff. eq. of the form $\frac{dy}{dx} = \frac{ax + by + c}{a'x + b'y + c'}$

If $\frac{a}{a'} \neq \frac{b}{b'}$ Then put $x = X + h,\ y = Y + k$

If $\frac{a}{a'} = \frac{b}{b'}$ Then put $ax + by = t,\ a + b\frac{dy}{dx} = \frac{dt}{dx}$

Q 18. Is the differential equation $\left(y^2 e^{xy^2} + 4x^3\right)dx + \left(2xy\, e^{xy^2} - 3y^2\right)dy = 0$ **exact?** (PTU, Dec. [illegible])

Solution. Compare the given diff. eq. with Mdx + Ndy = 0

Here

$$M = y^2 e^{xy^2} + 4x^3; \qquad N = 2xy\, e^{xy^2} - 3y^2$$

$$\frac{\partial M}{\partial y} = y^2 e^{xy^2}(2xy) + e^{xy^2}\,2y = e^{xy^2}(2y + 2xy^3) \qquad \frac{\partial N}{\partial x} = 2y\left[e^{xy^2} + x e^{xy^2} y^2\right] = e^{xy^2}(2y + 2xy^3)$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow$$ The given eq. is exact and its sol. is given by

$$\int_{y\ \text{const}} \left(y^2 e^{xy^2} + 4x^3\right)dx + \int -3y^2\, dy = c$$

$$y^2 \frac{e^{xy^2}}{y^2} + x^4 + (-y^3) = c$$

$\Rightarrow$ $e^{xy^2} + x^4 - y^3 = c$ is the req. sol.

Q 19. Solve :
$(2x^2y^2 + y)\,dx = (x^3y - 3x)\,dy$. (PTU, Dec. [illegible])

Ans. Compare the given diff. eq. with Mdx + Ndy = 0

$$M = 2x^2y^2 + y;\ N = -x^3y + 3x$$

$$\frac{\partial M}{\partial y} = 4x^2y + 1;\ \frac{\partial N}{\partial x} = -3x^2y + 3 \Rightarrow \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

The given diff. eq. is not exact.
The given diff. eq. can be written as
$x^2y(2y\,dx - x\,dy) + (y\,dx + 3x\,dy) = 0$
compare with $x^a y^b(my\,dx + nx\,dy) + x^{a'} y^{b'}(m'y\,dx + n'x\,dy) = 0$
$a = 2, b = 1, m = 2, n = -1, a' = b' = 0, m' = 1, n' = 3$

find h and k so that $\frac{a+h+1}{m} = \frac{b+k+1}{n}$ and $\frac{a'+h+1}{m'} = \frac{b'+k+1}{n'}$

i.e. $\frac{2+h+1}{2} = \frac{1+k+1}{-1} \Rightarrow h + 2k = -7$ —(1)

i.e. $\frac{0+h+1}{1} = \frac{0+k+1}{3} \Rightarrow 3h - k = -2$ —(2)

On solving (1) and (2), we get $h = \frac{-11}{7}$, $k = \frac{-19}{7}$

$$\text{I.F.} = x^h y^k = x^{\frac{-11}{7}} y^{\frac{-19}{7}}$$

Multiplying the given eq. by $x^{\frac{-11}{7}} y^{\frac{-19}{7}}$ and its sol. is given by

$$\int x^{\frac{-11}{7}} y^{\frac{-19}{7}}\left(2x^2y^2 + y\right)dx = c$$

$$\Rightarrow 2y^{\frac{-5}{7}} x^{\frac{10}{7}} \cdot \frac{7}{10} + y^{\frac{-12}{7}} x^{\frac{-4}{7}}\left(\frac{-7}{4}\right) = c$$

$$\Rightarrow 4y^{\frac{-5}{7}} x^{\frac{10}{7}} - 5y^{\frac{-12}{7}} x^{\frac{-4}{7}} = A, \text{ where } A = \frac{20c}{7}$$

Q 20. Define the Clairaut's equation and solve the differential equation, p = log (px−y). (PTU, Dec. 2010, 2011; May [illegible])

Ans. The clairaut's equation is of the form by y = px + f(p)
then its solution can be obtained by replacing p by constant c
i.e. $y = cx + f(c)$
Now $p = \log(px - y)$
$\Rightarrow$ $e^p = px - y \Rightarrow y = px - e^p$ It is of clairaut's form
Its solution is obtained by replacing p by c
i.e. $y = cx - e^c$ is the req. solution.

Q 21. Solve the differential equation
$(\sec x \tan x \tan y - e^x)\,dx + \sec x \sec^2 y\, dy = 0$ (PTU, Dec. [illegible])

Ans. Compare the given diff. eq. with Mdx + Ndy = 0

$$M = \sec x \tan x \tan y - e^x;\ N = \sec x \sec^2 y$$

$$\frac{\partial M}{\partial y} = \sec x \tan x \sec^2 y,\ \frac{\partial N}{\partial x} = \sec x \tan x \sec^2 y$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$ The given diff. eq. is exact and its solution is given by

$$\int_{y = \text{constant}} \left(\sec x \tan x \tan y - e^x\right)dx + \int 0\, dy = C$$

$\Rightarrow$ $\tan y \sec x - e^x = C$ is the required solution.

Q 22. Find the General Solution of the differential equation
$(2xy + x^2)\,y' = 3y^2 + 2xy$ (PTU, May [illegible])

Ans. The given diff. eq. can be written as

Now $e^{\int f(x)dx}$ be an integrating factor of eq (1) if

$M e^{\int f(x)dx} dx + N e^{\int f(x)dx} dy = 0$ is an exact diff. eq.

i.e. if $\frac{\partial}{\partial y}\left(M e^{\int f(x)dx}\right) = \frac{\partial}{\partial x}\left(N e^{\int f(x)dx}\right)$

i.e. if $\frac{\partial M}{\partial y} e^{\int f(x)dx} = \frac{\partial N}{\partial x} e^{\int f(x)dx} + N\, e^{\int f(x)dx} f(x)$

i.e. if $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = f(x)$ which is true as it is given

Hence $e^{\int f(x)dx}$ is an I.F. of eq (1)

Q 12. Find differential equation of S.H.M. given by x = Acos (nt + α), where n is constant. **(PTU, Dec. [illegible])**

Ans. $x = A\cos(nt + \alpha)$, Here A, α are arbitrary constants

$\Rightarrow \quad \frac{dx}{dt} = -A\sin(nt + \alpha).n$

$\Rightarrow \quad \frac{d^2x}{dt^2} = -An^2\cos(nt + \alpha) \Rightarrow \frac{d^2x}{dt^2} + n^2x = 0$

Q 13. Solve p = sin (y – xp). **(PTU, Dec. [illegible])**

Ans. $p = \sin(y - xp) \Rightarrow \sin^{-1}p = y - xp \Rightarrow y = px + \sin^{-1}p$

which is of clairaut's form and its solution is given by replacing p by constant C

i.e. $y = Cx + \sin^{-1}C$

Q 14. Solve $x\frac{dy}{dx} + y = x^3y^6$. **(PTU, Dec. [illegible])**

Ans. $x\frac{dy}{dx} + y = x^3y^6$, Dividing throughout by y^6

$\frac{1}{y^6}\frac{dy}{dx} + \frac{1}{y^5}\cdot\frac{1}{x} = x^2$; put $\frac{1}{y^5} = t \Rightarrow \frac{-5}{y^6}\frac{dy}{dx} = \frac{dt}{dx}$

$\Rightarrow \quad \frac{-1}{5}\frac{dt}{dx} + \frac{t}{x} = x^2 \Rightarrow \frac{dt}{dx} - \frac{5}{x}t = -5x^2$ which is linear diff. eq in t

$I.F = e^{\int\frac{-5}{x}dx} = e^{-5\log|x|} = \frac{1}{x^5}$

and solution is given by

$t\cdot\frac{1}{x^5} = \int -5x^2\cdot\frac{1}{x^5}dx + C$

$\Rightarrow \quad \frac{1}{y^5}\cdot\frac{1}{x^5} = -5\frac{x^{-3+1}}{-3+1} + C$

i.e. $\quad \frac{1}{x^5y^5} = \frac{5}{2x^2} + C$

Q 15. Solve clairaut's equation y = px + f (p) **(PTU, May [illegible])**

Solution.

It can be solvable for 'y' $\quad y = px + f(p) \quad$ ...(1)

Diff. w.r.t. x on both sides, we get

$p = \frac{dy}{dx} = p + x\frac{dp}{dx} + f'(p)\frac{dp}{dx} \Rightarrow (x + f'(p))\frac{dp}{dx} = 0$

$\Rightarrow \quad \frac{dp}{dx} = 0$; on Integrating we get

$p = \text{constant} = c$

∴ eq (1) gives

$y = cx + f(c)$ is the required solution.

Q 16. Check the equation $(3x^2 + 2e^y)\,dx + (2xe^y + 3y^2)\,dy = 0$ for exactness. **(PTU, Dec. [illegible])**

Solution. The given diff. eq be

$(3x^2 + 2e^y)dx + (2xe^y + 3y^2)dy = 0$ compare it with $Mdx + Ndy = 0$

Here $M = 3x^2 + 2e^y$; $N = 2xe^y + 3y^2$

$\frac{\partial M}{\partial y} = 2e^y$; $\frac{\partial N}{\partial x} = 2e^y$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ ∴ The given diff eq. be exact.

Q 17. Find solution of the differential equation $y' + y = y^2$. **(PTU, May [illegible])**

Solution. The given differential equation be

$y' + y = y^2 \Rightarrow \frac{dy}{dx} = y^2 - y$

on separation the variables, we get

$\frac{dy}{y(y-1)} = dx \Rightarrow \left[\frac{-1}{y} + \frac{1}{y-1}\right]dy = dx$

on integrating, we get

$-\log|y| + \log|y - 1| = x + c$

$\Rightarrow \quad \frac{y-1}{y} = Ae^x$ is the required solution.

$\Rightarrow \quad \left|y + \sqrt{x^2 + y^2}\right| = cx^2$

(iii) The given diff. eq. can be written as $y = px + p^2$, $p = \frac{dy}{dx}$ which is of clairaut's form. Its solution is given by putting p by constant c i.e. $y = cx + c^2$ is the required solution.

Q 6. Define Leibnitz's linear and Bernoulli's equations. (PTU, May 2007)
Solution. Linear differential Equation of Ist order

Its general form is $\frac{dy}{dx} + Py = Q$ where P, Q are functions of x alone.

Its Integrating factor $= e^{\int P\,dx}$
and hence solution is given by

$$y.e^{\int P\,dx} = \int Q.e^{\int P\,dx}\,dx + c$$

Similarly we can solve $\frac{dx}{dy} + Px = Q$ where P, Q are functions of y alone.

Hence sol. is given by

$$x e^{\int P\,dy} = \int Q e^{\int P\,dy}\,dy + c$$

Equations reducible to leibnitz's linear form (Bernoulli's form) :

An equation of the form $\frac{dy}{dx} + Py = Qy^n$ ...(1)

where P, Q are function of x alone is called Bernoulli's equation.
Dividing both sides of eq. (1) by y^n

$$\frac{1}{y^n}\frac{dy}{dx} + \frac{1}{y^{n-1}}P = Q \Rightarrow y^{-n}\frac{dy}{dx} + y^{1-n}P = Q$$

putting $\quad y^{1-n} = z \Rightarrow (1-n)y^{-n} = \frac{dz}{dy} \Rightarrow (1-n)y^{-n}\frac{dy}{dx} = \frac{dz}{dx}$

$\Rightarrow \quad \frac{1}{1-n}\frac{dz}{dx} + Pz = Q$

$\Rightarrow \quad \frac{dz}{dx} + (1-n)Pz = (1-n)Q$ which is of linear's form.

Then apply the procedure as for linear eqn.

Q 7. Solve $(x^2 - ay)\,dx = (ax - y^2)\,dy$. (PTU, May 2005)
Ans. Compare $(x^2-ay)dx-(ax-y^2)dy = 0$ with $Mdx + Ndy = 0$

$M = x^2 - ay ; N = y^2 - ax$

$\frac{\partial M}{\partial y} = -a ; \frac{\partial N}{\partial x} = -a \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

The given eq. is exact and its solution is given by

$$\int (x^2 - ay)dx + \int_{x=\text{const}} y^2 dy = 0 \Rightarrow \frac{x^3}{3} - axy + \frac{y^3}{3} = C$$

Q 8. When a solution of a differential equation is called its general solution? (PTU, Dec. 2005)

Ans. A solution of the differential equation in which the number of independent arbitrary constants is same as order of differential equation. It is also called complete solution.

e.g. The given differential equation be, $\frac{d^2y}{dx^2} - y = 0$ i.e. $(D^2 - 1)y = 0$

General solution $= y = C_1e^x + C_2e^{-x}$

Here the number of arbitary constants = 2 = order of differential equation.

Q 9. Solve $\frac{dy}{dx} = \frac{y}{x}$ (PTU, Dec. 2005)

Ans. $\frac{dy}{dx} = \frac{y}{x} \Rightarrow \frac{dy}{y} = \frac{dx}{x}$ on integrating we get

$\Rightarrow \quad$ Log y = Log x + Log c $\Rightarrow$ y = cx is the required solution.

Q 10. Define an integrating factor. Find the integrating factor of the differential equation $(y-1)\,dx - x\,dy = 0$. (PTU, May 2006)

Ans.
$\mu = \mu(x, y)$ is called I.F. of diff. eq. $Mdx + Ndy = 0$
if $\quad \mu Mdx + \mu Ndy = 0$ is exact there exists $F = F(x, y)$
s.t. $\quad \mu Mdx + \mu Ndy = dF$

The given diff. eq. be $ydx - xdy - dx = 0$ ...(1)

Multiply eq (1) by $\frac{1}{x^2}$, we get

$\Rightarrow \quad d\left(-\frac{y}{x}\right) + d\left(\frac{1}{x}\right) = 0$ The given eq (1) is exact

and $\quad$ I.F. $= \frac{1}{x^2}$

Q 11. If $\left[\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)/N\right] = f(x)$ a function of x alone, there show that $\int_e f(x)dx$ is an integrating factor of

$M(x, y)\,dx + N(x, y)\,dy = 0$. (PTU, May 2006)

Ans. The given eq. $Mdx + Ndy = 0$ ...(1)

Q 30. Solve $x\frac{dy}{dx} - y = x\sqrt{x^2+y^2}$. (PTU, May 2003)

Ans. put $y = vx,\ \frac{dy}{dx} = v + x\frac{dv}{dx}$

$\Rightarrow \quad x\left[v + x\frac{dv}{dx}\right] - vx = x\sqrt{x^2+v^2x^2}$

$\Rightarrow \quad x^2\frac{dv}{dx} = x^2\sqrt{1+v^2} \Rightarrow \frac{1}{\sqrt{1+v^2}}dv = dx$, on integrating, we get

$\Rightarrow \quad \text{Log}\left|v+\sqrt{1+v^2}\right| = x + c$

$\Rightarrow \quad \text{Log}\left|\frac{y}{x}+\frac{1}{x}\sqrt{x^2+y^2}\right| = x + c$ is the req. sol.

Q 31. Solve : $\cos(x+y)\,dy = dx$ (PTU, May 2002)

Ans. $\cos(x+y)\,dy = dx$

put $x + y = t \Rightarrow 1 + \frac{dy}{dx} = \frac{dt}{dx}$

$\Rightarrow \quad \cos t\left[\frac{dt}{dx} - 1\right] = 1 \Rightarrow \frac{dt}{dx} - 1 = \frac{1}{\cos t} \Rightarrow \frac{dt}{dx} = \frac{1}{\cos t} + 1$

$\Rightarrow \quad \frac{\cos t\,dt}{1+\cos t} = dx$, on integrating, we get

$\Rightarrow \quad \int dt - \int \frac{1}{1+\cos t} \times \frac{1-\cos t}{1-\cos t}\,dt = x + c$

$\Rightarrow \quad t - \int\left(\text{cosec}^2 t - \cot t\,\text{cosec}\,t\right)dt = x + c$

$\Rightarrow \quad t + \cot t - \text{cosec}\,t = x + c$

$\Rightarrow \quad x + y + \cot(x+y) - \text{cosec}(x+y) = x + c$

$\Rightarrow \quad y + \cot(x+y) - \text{cosec}(x+y) = c$ is the req. sol.

Q 32. Solve the problem

$$\left(xy^2 - e^{x^{-3}}\right)dx - x^2y\,dy = 0$$

(PTU, May 2010 ; Dec. 2006, 2003)

Ans. Compare $\left(xy^2 - e^{x^{-3}}\right)dx - x^2y\,dy = 0$ ..... (1) with $Mdx + Ndy = 0$

Where $M = xy^2 - e^{x^{-3}}$; $N = -x^2y$

$\frac{\partial M}{\partial y} = 2xy$; $\frac{\partial N}{\partial x} = -2xy$; $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ ∴ eq (1) is not exact

Also $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{2xy + 2xy}{-x^2y} = \frac{-4xy}{x^2y} = \frac{-4}{x} = f(x)$

$\text{I.F.} = e^{\int -\frac{4}{x}dx} = e^{-4\log x} = \frac{1}{x^4}$

and its solution is given by

$\int \frac{xy^2 - e^{x^{-3}}}{x^4}dx + 0 = C$

$\Rightarrow \quad \int \frac{1}{x^3}y^2\,dx - \int e^{x^{-3}}\frac{1}{x^4}dx = C$

$\Rightarrow \frac{-y^2}{2x^2} + \frac{1}{3}\int e^t\,dt = C$; put $\frac{1}{x^3} = t \Rightarrow \frac{-3}{x^4}dx = dt$

$\Rightarrow \frac{-y^2}{2x^2} + \frac{1}{3}e^{x^{-3}} = C$ is the required solution.

Q 33. Solve $(x^2y - 2xy^2)\,dx - (x^3 - 3x^2y)\,dy = 0$ (PTU, Dec. 2010, 2003)

Ans. Compare the given diff. eq. with $Mdx + Ndy = 0$

where $M = x^2y - 2xy^2$; $N = -x^3 + 3x^2y$

$\frac{\partial M}{\partial y} = x^2 - 4xy$; $\frac{\partial N}{\partial x} = -3x^2 + 6xy$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ ∴ The given diff. eq. is not exact

Now I.F. $= \frac{1}{Mx + Ny} = \frac{1}{x^3y - 2x^2y^2 - x^3y + 3x^2y^2} = \frac{1}{x^2y^2}$

[M, N are Homogeneous function of x and y]

Multiply given eq. by $\frac{1}{x^2y^2}$, we get

$$\text{I.F.} = e^{\int -1\,dx} = e^{-x} \text{ and solution is given by}$$

$\Rightarrow$ $y\,e^{-x} = \int e^{-x}\,x\,dx + c_2$

$\Rightarrow$ $y = [-xe^{-x} - e^{-x}] + c_2$

$\Rightarrow$ $y = -(x+1) + c_2 e^{x}$ ...(4)

$\therefore$ General solution is given by

$$\left(y - \frac{x^2}{2} - c_1\right)(y + x + 1 - c_2 e^{x}) = 0$$

Q 26. Obtain the general and as well as singular solution of the non-linear equation $y = xy' + (y')^2$. **(PTU, Dec. 2007)**

Solution. The given diff. eq can be written as $y = xp + p^2$; where $p = y' = \frac{dy}{dx}$ ...(1)

Diff. (1) both sides w.r.t. x, we get

$$p = x\frac{dp}{dx} + p + 2p\frac{dp}{dx} \Rightarrow (x + 2p)\frac{dp}{dx} = 0 \Rightarrow \frac{dp}{dx} = 0 \text{ or } x + 2p = 0$$

$\Rightarrow$ $p = c$ $\therefore$ eq (2) gives $y = cx + c^2$ ...(2)

$\therefore$ eq (2) gives the general solution

Diff. (2) both sides w.r.t. 'c' we get

$0 = x + 2c$ ...(3)

To find the singular solution we have to eliminate c from (2) and (3)

i.e. $$y = x\left(-\frac{x}{2}\right) + \left(-\frac{x}{2}\right)^2 \Rightarrow y = \frac{-x^2}{2} + \frac{x^2}{4} = \frac{-x^2}{4}$$

is the req. singular solution.

Q 27. Solve the initial value problem $e^x(\cos y\,dx - \sin y\,dy) = 0$, $y(0) = 0$. **(PTU, May 2008)**

Solution. The given diff. eq. be $e^x(\cos y\,dx - \sin y\,dy) = 0$, $y(0) = 0$

$\Rightarrow$ $\cos y\,dx + \sin y\,dy = 0$

$\Rightarrow$ $dx + \frac{\sin y}{\cos y}dy = 0$; on integrating, we get

$\Rightarrow$ $x + \log|\sec y| = c$ ...(1)

also $y(0) = 0$ then eq (1) gives; $0 + c = c$

$\therefore$ eq (1) gives;

$x + \log|\sec y| = 0$

Q 28. Solve $(xy^3 + y)\,dx + 2(x^2y^2 + x + y^4)\,dy = 0$. **(PTU, Dec. 2020, 2011, 2008)**

Solution. The given diff. eq. be

$(xy^3 + y)\,dx + 2(x^2y^2 + x + y^4)\,dy = 0$ ...(1)

Compare eq (1) with $Mdx + Ndy = 0$

Here $M = xy^3 + y$; $N = 2(x^2y^2 + x + y^4)$

i.e. $$\frac{\partial M}{\partial y} = 3xy^2 + 1;\ \frac{\partial N}{\partial x} = 2(2xy^2 + 1)$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \text{ hence eq (1) is not exact}$$

Now $$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M} = \frac{-xy^2 - 1}{y(1 + xy^2)} = -\frac{1}{y} = g(y)$$

$$\text{I.F.} = e^{\int -g(y)\,dy} = e^{\int \frac{1}{y}dy} = y$$

Multiply eq (1) by y, we get

$y(xy^3 + y)\,dx + 2y(x^2y^2 + x + y^4)\,dy = 0$ ...(2)

$\therefore$ eq (2) becomes exact and its solution is given by

$$\int_{y\text{-constant}} (xy^4 + y^2)\,dx + 2\int y^5\,dy = 0$$

$$y\left[\frac{x^2y^3}{2} + yx\right] + \frac{1}{3}y^6 = C \text{ is the required solution}$$

Q 29. Solve $\frac{dx}{y+z} = \frac{dy}{z+x} = \frac{dz}{x+y}$ **(PTU, Dec. 2002)**

Ans. $$\frac{dx}{y+z} = \frac{dy}{z+x} = \frac{dz}{x+y} = \frac{dx - dy}{y - x} = \frac{dy - dz}{z - y} = \frac{dz - dx}{x - z}$$

from (4) and (5) fraction

$$\frac{dx - dy}{x - y} = \frac{dy - dz}{y - z} \text{ on integrating we get}$$

$$\text{Log}\frac{x - y}{y - z} = \text{Log}\,C_1 \Rightarrow \frac{x - y}{y - z} = C_1$$

from (5) and (6) fraction, we get

$$\frac{dy - dz}{y - z} = \frac{dz - dx}{z - x} \text{ on integrating we get}$$

$$\log\frac{y - z}{z - x} = \log C_2 \Rightarrow \frac{y - z}{z - x} = C_2$$

and its general sol. is given by $\phi(C_1, C_2) = 0$

Where C_1, C_2 are arbitary constants

$(2xy + x^2)\frac{dy}{dx} = (3y^2 + 2xy) \Rightarrow (3y^2 + 2xy)\,dx - (2xy + x^2)\,dy = 0$ ...(1)

Compare with $Mdx + Ndy = 0$, $M = 3y^2 + 2xy$; $N = -2xy - x^2$

$$\frac{\partial M}{\partial y} = 6y + 2x \ ; \ \frac{\partial N}{\partial x} = -2y - 2x \qquad \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

eq (1) is not exact but M, N are homogeneous function of x and y

$$\text{I.F.} = \frac{1}{Mx + Ny} = \frac{1}{3y^2x + 2x^2y - 2xy^2 - x^2y}$$

$$= \frac{1}{xy^2 + x^2y}$$

Multiply eq (1) by $\frac{1}{xy^2 + x^2y}$ and its solution is given by

$$\int \frac{(3y^2 + 2xy)\,dx}{xy^2 + x^2y} + 0 = C$$

$$\Rightarrow \quad \int \frac{(3y + 2x)\,dx}{x(x + y)} = \log C \Rightarrow \int \left[\frac{3}{x} - \frac{1}{x + y}\right] dx = \log C$$

$\Rightarrow 3\log|x| - \text{Log}\,|x + y| = \text{Log}\,C \Rightarrow \text{Log}\,\frac{x^3}{x + y} = \text{Log}\,C \Rightarrow x^3 = C(x + y)$ be the req. sol.

Q 23. Solve $\frac{dy}{dx} = \sin(x + y)$ (PTU, May 2006)

Ans. Put $x + y = t \Rightarrow 1 + \frac{dy}{dx} = \frac{dt}{dx}$

The given diff. eq. gives

$$\frac{dt}{dx} - 1 = \sin t \Rightarrow \frac{dt}{dx} = 1 + \sin t \Rightarrow \frac{1}{1 + \sin t}\,dt = dx$$

On integrating, we get

$$\int \frac{1}{1 + \sin t} \cdot \frac{1 - \sin t}{1 - \sin t}\,dt = \int dx + C$$

$$\Rightarrow \quad \int \frac{1 - \sin t}{\cos^2 t}\,dt = x + c$$

$\Rightarrow \quad \tan t - \sec t = x + c \Rightarrow \tan(x + y) - \sec(x + y) = x + c$ is the req. sol.

Q 24. Solve $(3xy^2 - y^3)\,dx - (2x^2y - xy^2)\,dy = 0$ (PTU, May 2007)

Solution. Here $M = 3xy^2 - y^3$; $N = -(2x^2y - xy^2)$

$$\frac{\partial M}{\partial y} = 6xy - 3y^2 \ ; \qquad \frac{\partial N}{\partial x} = -4xy + y^2$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

∴ The given eq. is not exact.

Here M, N are both homogeneous function of x and y

$$\text{I.F.} = \frac{1}{Mx + Ny} = \frac{1}{3x^2y^2 - xy^3 - 2x^2y^2 + xy^3}$$

$$= \frac{1}{x^2y^2}$$

Multiply the given eq. by $\frac{1}{x^2y^2}$, we get

$$\frac{1}{x^2y^2}(3xy^2 - y^3)\,dx - \frac{1}{x^2y^2}(2x^2y - xy^2)\,dy = 0$$

∴ The given eq. becomes an exact diff. eq. and its solution is

$$\int \frac{1}{x^2y^2}(3xy^2 - y^3)\,dx - \int \frac{2}{y}\,dy = 0$$

$$3\log x + \frac{y}{x} - 2\log y = c$$

Q 25. Solve $p(p - y) = x(x + y)$ (PTU, May 2007)

Solution. The given diff. eq. be

$$p(p - y) = x(x + y)$$

$\Rightarrow \quad p^2 - x^2 - py - xy = 0 \Rightarrow (p - x)(p + x) - y(p + x) = 0$

$\Rightarrow \quad (p + x)(p - x - y) = 0 \Rightarrow \quad (p + x)(p - x - y) = 0$

Its components eqs are

$p + x = 0$ ...(1)

and $p - x - y = 0$ ...(2)

From (1) ; $p + x = 0 \Rightarrow p = -x \Rightarrow \frac{dy}{dx} = -x$

on integrating, we get

$$y = -\frac{x^2}{2} + c_1 \qquad \text{...(3)}$$

From (2) ; we have $\frac{dy}{dx} - y = x$; It is linear diff. eq in 'y', we get ball of mass $m_1 = h_1 = \frac{V_1^2 \sin^2\theta}{2g}$ ...(1)

Similarly maximum height attained by 2nd ball of mass m_2

$$= h_2 = \frac{V_2^2 \sin^2\theta}{2g}$$ ...(2)

Now $V_1 = 2V_2$

$$h_1 = \frac{4\,V_2^2 \sin^2\theta}{2g}$$ ...(3)

and $h_2 = \frac{V_2^2 \sin^2\theta}{2g}$ ...(4)

on dividing (3) and (4); we have

$$h_1 = 4h_2$$

Q 41. Solve the following :

(a) $xy\,(1 + xy^2)\,\frac{dy}{dx} = 1$

(b) $\frac{dy}{dx} = \frac{-\left(3x^2 + 6xy^2\right)}{6x^2y + 4y^3}$

(c) $(px - y)\,(x + py) = 2p.$ **(PTU, May 2008)**

Solution. (a) The given equation can be written as

$$\frac{dx}{dy} - yx = y^3 x^2$$

Divide by x^2, we have $x^{-2}\frac{dx}{dy} - yx^{-1} = y^3$ ..... (1)

Putting $x^{-1} = z$ so that $-x^{-2}\frac{dx}{dy} = \frac{dz}{dy}$

$$x^{-2}\frac{dx}{dy} = -\frac{dz}{dy}$$

equation (1) becomes,

$$-\frac{dz}{dy} - yz = y^3$$

$$\frac{dz}{dy} + yz = -y^3 \text{; which is linear in } z$$

$$I.F. = e^{\int y\,dy} = e^{\frac{1}{2}y^2}$$

The solution is $z\,(I.F.) = \int -y^3\,(I.F.)\,dy + C$

$$z\,e^{\frac{1}{2}y^2} = \int -y^3 e^{\frac{1}{2}y^2}\,dy + C$$

$$z\,e^{\frac{1}{2}y^2} = -\int y^2 e^{\frac{1}{2}y^2}\,y\,dy + C$$

$$= -\int 2t\,e^t\,dt + C \text{; where } t = \frac{1}{2}y^2 \Rightarrow dt = y\,dy$$

$$z\,e^{\frac{1}{2}y^2} = -2\left[te^t - \int 1.e^t\,dt\right] + C = -2\,(te^t - e^t) + C$$

$$= -2\,e^{\frac{1}{2}y^2}\left(\frac{1}{2}y^2 - 1\right) + C$$

$$z = -2\left(\frac{1}{2}y^2 - 1\right) + Ce^{-\frac{1}{2}y^2}$$

$$\frac{1}{x} = 2 - y^2 + Ce^{-\frac{1}{2}y^2}$$

(b) $(3x^2 + 6xy^2)\,dx + (6x^2y + 4y^3)\,dy = 0$

Comparing the given differential equation with

$$M\,dx + N\,dy = 0$$

Here $M = 3x^2 + 6xy^2,\ N = 6x^2y + 4y^3$

$$\frac{\partial M}{\partial y} = 12xy,\ \frac{\partial N}{\partial x} = 12xy$$

$\Rightarrow$ $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

∴ The given differential equation is exact and hence the solution is given by

$$\int_{y\,\text{const}} M\,dx + \int (\text{Terms of } N \text{ not containing } x)\,dy = C$$

$$\int_{y\,\text{const}} \left(3x^2 + 6xy^2\right)dx + \int 4y^3\,dy = C$$

$$\frac{3x^3}{3} + 6y^2\,\frac{x^2}{2} + \frac{4y^4}{4} = C$$

$x^3 + 3x^2y^2 + y^4 = C$, is the required solution.

(c) $(px - y)(py + x) = 2p$. Let $X = x^2$ and $Y = y^2$

$$dX = 2x\,dx,\ dY = 2y\,dy$$

$$I.F. = e^{\int 2x\,dx} = e^{x^2}$$

and its solution is given by

$$y \cdot e^{x^2} = \int 2e^{-x^2} \cdot e^{x^2}\,dx + C \Rightarrow y\,e^{x^2} = 2x + C$$

(b) $$\frac{dy}{dx} - y\tan x = -y^2 \sec x \Rightarrow \frac{1}{y^2}\frac{dy}{dx} - \frac{1}{y}\tan x = -\sec x \quad \ldots(1)$$

$$\text{put } \frac{1}{y} = z \Rightarrow -\frac{1}{y^2}\frac{dy}{dx} = \frac{dz}{dx} \therefore \text{ eq (1) gives}$$

$$\Rightarrow \quad \frac{-dz}{dx} - (\tan x)\,z = -\sec x \Rightarrow \frac{dz}{dx} + (\tan x)\,z = \sec x$$

which is linear diff. eq. of first order

$$\text{and } I.F. = e^{\int \tan x\,dx} = e^{-\log|\cos x|} = \sec x$$

and solution is given by

$$y\sec x = \int \sec^2 x\,dx + c \Rightarrow y\sec x = \tan x + C$$

(c) $(2x\log x - xy)\,dy + 2y\,dx = 0 \quad \ldots(1)$ Compare with $Mdx + Ndy = 0$

$M = 2y$; $N = 2x\log x - xy$

$$\frac{\partial M}{\partial y} = 2 ; \frac{\partial N}{\partial x} = 2\left[x \cdot \frac{1}{x} + \log x\right] - y$$

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = -2\log x + y \text{ and } \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{-(2\log x - y)}{x(2\log x - y)} = \frac{-1}{x}$$

$$I.F. = e^{\int -\frac{1}{x}dx} = e^{-\log|x|} = \frac{1}{x}$$

Multiply eq (1) by $\frac{1}{x}$, we get

$\frac{2y}{x}dx + (2\log x - y)\,dy = 0$ and its solution is given by

$$\int \frac{2y}{x}dx + \int -y\,dy = C \Rightarrow 2y\log|x| - \frac{y^2}{2} = C$$

Q 38. If $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then prove that the differential equation $M(x, y)\,dx + N(x, y)\,dy = 0$ is exact. (PTU, May 2009)

Ans. As given $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ We want to prove that $Mdx + Ndy = 0$ is exact

$$\text{Let } F = \int_{y\text{-const}} M\,dx \Rightarrow \frac{\partial F}{\partial x} = M \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial^2 F}{\partial y \partial x}$$

$$\text{again also } \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = \frac{\partial^2 F}{\partial x \partial y} = \frac{\partial^2 F}{\partial y \partial x}$$

$$\Rightarrow \quad \frac{\partial N}{\partial x} = \frac{\partial^2 F}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y}\right)$$

Integrate w.r.t. x (taking y as constant)

$$\Rightarrow \quad N = \frac{\partial F}{\partial y} + f(y)$$

$$\Rightarrow \quad Mdx + Ndy = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy + f(y)\,dy$$

$$= dF + f(y)\,dy = d(F + \phi(y))$$

$$\Rightarrow \quad Mdx + Ndy = 0 \text{ is exact.}$$

Q 39. Explain the technique of Bernoulli's linear equation. (PTU, Dec. 2008)

Solution. An equation of the form $\frac{dy}{dx} + Py = Qy^n \quad \ldots(1)$

where P, Q are function of x alone is called Bernoulli's equation.

Dividing both sides of eq. (1) by y^n; we get

$$\frac{1}{y^n}\frac{dy}{dx} + \frac{1}{y^{n-1}}P = Q \Rightarrow y^{-n}\frac{dy}{dx} + y^{1-n}P = Q$$

$$\text{putting} \quad y^{1-n} = z \Rightarrow (1-n)\,y^{-n} = \frac{dz}{dy} \Rightarrow (1-n)\,y^{-n}\frac{dy}{dx} = \frac{dz}{dx}$$

$$\Rightarrow \quad \frac{1}{1-n}\frac{dz}{dx} + Pz = Q$$

$$\Rightarrow \quad \frac{dz}{dx} + (1-n)Pz = (1-n)Q \text{ which is of linear's form.}$$

Then apply the procedure as for linear eqs. i.e. $I.F. = e^{\int (1-n)P\,dx}$

and its solution is given by

$$z \cdot e^{\int (1-n)P\,dx} = \int (1-n)\,Q \cdot e^{\int (1-n)P\,dx}\,dx + c$$

Q 40. Two balls of m_1 and m_2 gms are projected vertically upward such that the velocity of projection of m_1 is double that of m_2. If the maximum height to which m_1 and m_2 rise, be h_1 and h_2 respectively, then

(i) $h_1 = 2h_2$ (ii) $2h_1 = h_2$ (iii) $h_1 = 4h_2$ (iv) $4h_1 = h_2$ (PTU, May 2009)

Solution. Given $V_1 = 2V_2$ (Where V_1 is velocity of ball which has mass m_1 and V_2 is velocity of ball which has mass m_2) and maximum height attained by

$\frac{1}{x^2y^2}(x^2y - 2xy^2)\,dx - \frac{1}{x^2y^2}(x^3 - 3x^2y)\,dy = 0$ is exact and

Its solution is given by

$$\int \frac{1}{x^2y^2}\left(x^2y - 2xy^2\right)dx + \int \frac{3}{y}\,dy = c$$

$$\Rightarrow \quad \frac{x}{y} - 2\log|x| + 3\log|y| = c$$

Q 34. Solve $\left(x\tan\frac{y}{x} - y\sec^2\frac{y}{x}\right)dx + x\sec^2\frac{y}{x}\,dy = 0$ (PTU, Dec. 2003)

Ans. $\left(x\tan\frac{y}{x} - y\sec^2\frac{y}{x}\right)dx + x\sec^2\frac{y}{x}\,dy = 0$ The given eq. can be

written as $\frac{dy}{dx} = \frac{\left(x\tan\frac{y}{x} - y\sec^2\frac{y}{x}\right)}{-x\sec^2\frac{y}{x}}$

put $\frac{y}{x} = v \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

$$\Rightarrow v + x\frac{dv}{dx} = \frac{x\tan v - vx\sec^2 v}{-x\sec^2 v} \Rightarrow \frac{x\,dv}{dx} = \frac{x\tan v - vx\sec^2 v}{-x\sec^2 v} - v$$

$$\Rightarrow \frac{x\,dv}{dx} = \frac{x\tan v}{-x\sec^2 v} = \frac{-\tan v}{\sec^2 v} \Rightarrow \frac{\sec^2 v}{\tan v}\,dv = \frac{-dx}{x}$$

on integrating we get,

$$\Rightarrow \log|\tan v| = -\log x + \log c \Rightarrow \log\left|x\tan\frac{y}{x}\right| = \log c$$

$x\tan\frac{y}{x} = c$ is the required solution.

Q 35. Solve : $\frac{dy}{dx} = \frac{x+y}{x-y}$ (PTU, May 2004)

Ans. The given diff. eq. be $\frac{dy}{dx} = \frac{x+y}{x-y}$ It can be written as $(x+y)\,dx - (x-y)\,dy = 0$

Compare with $M\,dx + N\,dy = 0$ $M = x + y$; $N = -(x - y)$

$\frac{\partial M}{\partial y} = 1,\ \frac{\partial N}{\partial x} = -1 \quad \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

eq (1) is not exact also eq (1) is a homogeneous diff. eq.

$$\text{I.F.} = \frac{1}{Mx + Ny} = \frac{1}{(x+y)x - y(x-y)} = \frac{1}{x^2+y^2}$$

Multiply eq (1) by, we get

$\frac{x+y}{x^2+y^2}\,dx - \frac{x-y}{x^2+y^2}\,dy = 0$ and its solution is given by

$$\int \frac{x+y}{x^2+y^2}\,dx = C \Rightarrow \int \frac{x}{x^2+y^2}\,dx + y\int \frac{1}{x^2+y^2}\,dx = C$$

$$\Rightarrow \frac{1}{2}\log(x^2+y^2) + y\cdot\frac{1}{y}\tan^{-1}\frac{x}{y} = C \Rightarrow \frac{1}{2}\log(x^2+y^2) + \tan^{-1}\frac{x}{y} = C$$

Q 36. Solve : $e^y y' = e^x(e^x - e^y)$. (PTU, May 2004)

Ans. The given diff. eq. can be written as $e^y\frac{dy}{dx} = e^x(e^x - e^y)$ ...(1)

put $e^y = t \Rightarrow e^y\frac{dy}{dx} = \frac{dt}{dx}$ eq (1) gives

$$\frac{dt}{dx} = e^x(e^x - t) \Rightarrow \frac{dt}{dx} + e^x t = e^{2x}$$

which is linear diff. eq. of 1st order

$\text{I.F.} = e^{\int e^x dx} = e^{e^x}$ and its solution is given by

$$t\,e^{e^x} = \int e^{2x} e^{e^x}\,dx + C$$

put $e^x = z$, $e^x\,dx = dz$

$$t\,e^{e^x} = \int z\,e^z\,dz + C$$

$$t\,e^{e^x} = (e^x - 1)\,e^{e^x} + C$$

$$\Rightarrow \quad e^y e^{e^x} = (e^x - 1)\,e^{e^x} + C$$

Q 37. Solve any two of the following differential equations :

(a) $\frac{dy}{dx} + 2xy = 2e^{-x^2}$

(b) $\frac{dy}{dx} = y\tan x - y^2\sec x$

(c) $(2x\log x - xy)\,dy + 2y\,dx = 0$. (PTU, Dec. 2004)

Ans. (a) $\frac{dy}{dx} + 2xy = 2e^{-x^2}$ which is linear diff. eq. of 1st order

$$I.F. = e^{\int \frac{-5}{x}dx} = e^{-5\log x} = \frac{1}{x^5}$$

Hence, solution of given equation is

$$t\frac{1}{x^5} = \int -5x^2 \cdot \frac{1}{x^5}\, dx + c = \frac{-5}{-2x^2} + c$$

$$\frac{1}{x^5 y^3} = \frac{5}{2x^2} + c$$

Q 49. Solve :

(a) (y + x) dy = (y − x) dx

(b) (x − 2y + 1) dx + (4x − 3y − 6) dy = 0. (PTU, May 2011)

Solution. (a) Given differential equation can be written as $\frac{dy}{dx} = \frac{y-x}{y+x}$ ...(1)

Put $y = vx$

$\Rightarrow \quad \frac{dy}{dx} = v + x\frac{dv}{dx}$

Therefore, eqn. (1) gives,

$$v + x\frac{dv}{dx} = \frac{vx - x}{vx + x} = \frac{v-1}{v+1}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{v-1}{v+1} - v = \frac{v - 1 - v^2 - v}{v+1}$$

$$x\frac{dv}{dx} = -\frac{1+v^2}{1+v}$$

$$\Rightarrow \quad \frac{(1+v)}{1+v^2}\, dv = -\frac{dx}{x}$$

On integrating, we get

$$\Rightarrow \tan^{-1} v - \frac{1}{2}\log\left(1+v^2\right) = -\log x + c$$

$$\Rightarrow \tan^{-1}\frac{y}{x} - \frac{1}{2}\log\left(1+\frac{y^2}{x^2}\right) + \log x = c$$

$$\Rightarrow 2\tan^{-1}\frac{y}{x} - \log\left(\frac{x^2+y^2}{x^4}\right) = C$$

where $C = 2c$ is the required solution.

(b) $$\frac{dy}{dx} = -\frac{x - 2y + 1}{4x - 3y - 6} \qquad \left[\because \frac{a}{a'} \neq \frac{b}{b'}\right]$$

put $x = X + h$; $y = Y + k$, $dx = dX$; $dy = dY$

Thus given eqn becomes,

$$\frac{dY}{dX} = -\frac{X - 2Y + h - 2k + 1}{4X - 3Y + 4h - 3k - 6} \quad ...(1)$$

Choosing h, k so that $h - 2k + 1 = 0$; $4h - 3k - 6 = 0$

on solving $h = 3$; $k = 2$

∴ eqn (1) gives;

$$\frac{dY}{dX} = -\frac{X - 2Y}{4X - 3Y}. \text{ Put } Y = vX \Rightarrow \frac{dY}{dX} = v + X\frac{dv}{dX}$$

$$v + X\frac{dv}{dX} = -\frac{X - 2vX}{4X - 3vX} = -\frac{1-2v}{4-3v}$$

$$X\frac{dv}{dX} = \frac{-1+2v}{4-3v} - v = \frac{3v^2 - 2v - 1}{4-3v}$$

$$\Rightarrow \quad \frac{(4-3v)\,dv}{3v^2-2v-1} = \frac{dX}{X} \Rightarrow \int \frac{(3v-4)\,dv}{3v^2-2v-1} = -\int\frac{dX}{X} + \log C$$

$$\Rightarrow \frac{1}{2}\left[\log\left|3v^2-2v-1\right| - 6\int\frac{dv}{3v^2-2v-1}\right] = -\int\frac{dX}{X} + \frac{1}{4}\log C$$

$$\Rightarrow \frac{1}{2}\left[\log\left|3v^2-2v-1\right| - \frac{6}{3}\int\frac{dv}{v^2 - \frac{2}{3}v + \frac{1}{9} - \frac{1}{9} - \frac{1}{3}}\right] = -\int\frac{dX}{X} + \frac{1}{4}\log C$$

$$\Rightarrow \frac{1}{2}\log\left|3v^2-2v-1\right| - \int\frac{dv}{\left(v-\frac{1}{3}\right)^2 - \left(\frac{2}{3}\right)^2} = -\int\frac{dX}{X} + \frac{1}{4}\log C$$

$$\Rightarrow \frac{1}{2}\log\left|3v^2-2v-1\right| - \frac{3}{4}\log\left|\frac{3v-3}{3v+1}\right| = -\log X + \frac{1}{4}\log C$$

$$\Rightarrow \log\frac{\left|3v^2-2v-1\right|^2}{(3v-3)^3}(3v+1)^3 X^4 = \log C$$

$$\Rightarrow \frac{\left(3Y^2 - 2XY - X^2\right)^2 (3Y+X)^3}{(3Y-3X)^3} = C$$

$\Rightarrow \quad (3Y + X)^2 = 27C(Y - X) \Rightarrow (x + 3y - 9)^2 = A(y - x + 1)$ is the required solution.

Q 50. Solve : $xp^2 - yp - y = 0$ (PTU, May 2011)

Solution. Given diff. eqn. be,

$$= \left(\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy\right) + f(y)\,dy$$

$$Mdx + Ndy = du + f(y)\,dy$$

Which is an exact differential.

$\because$ f(y) dy is an exact differential as $f(y)\,dy = d\left[\int f(y)\,dy\right]$

$$Mdx + Ndy = 0 \text{ is exact.}$$

Condition is sufficient.

Q 45. Solve : xdy – ydx = $(x^2 + y^2)$ dx. (PTU, Dec. 200...)

Solution. We solve it by inspection method. Write the given differential equation as

$$\frac{x\,dy - y\,dx}{x^2 + y^2} - dx = 0$$

$\Rightarrow d\left(\tan^{-1}\frac{y}{x}\right) - dx = 0$, integrating, we get

$$\because \left[d\left(\tan^{-1}\frac{y}{x}\right) = \frac{1}{1+\frac{y^2}{x^2}}\,d\left(\frac{y}{x}\right) = \frac{x^2}{x^2+y^2}\cdot\frac{xdy - ydx}{x^2} = \frac{xdy - ydx}{x^2+y^2}\right]$$

$\tan^{-1}\frac{y}{x} - x = c$ is the required solution.

Q 46. For what value of 'k' the differential equation

$\left(1 + e^{kx/y}\right)dx + e^{x/y}\left(1 - \frac{x}{y}\right)dy = 0$ **is exact.** (PTU, May 2010)

Solution. On comparing given diff. equ. with Mdx + Ndy = 0

Here $M = \left(1 + e^{kx/y}\right)$; $N = \left(1 - \frac{x}{y}\right)$

$$\frac{\partial M}{\partial y} = e^{kx/y}\left(\frac{-x}{y^2}\right);\ \frac{\partial N}{\partial x} = e^{x/y}\left(-\frac{x}{y^2}\right)$$

The given diff. equ. is exact if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

i.e. $e^{kx/y}\left(\frac{-x}{y^2}\right) = e^{x/y}\left(-\frac{x}{y^2}\right) \Rightarrow k = 1.$

Q 47. Find the solution of the equation y – 2px = $\tan^{-1}(xp^2)$ where p = $\frac{dy}{dx}$. (PTU, May 2010)

Solution. Given $y = 2px + \tan^{-1}(xp^2)$ ...(1)

Differentiate eqn. (1) w.r.t. x, we get

$$p = 2\left[p + x\frac{dp}{dx}\right] + \frac{1}{1+x^2p^4}\left[p^2 + 2xp\frac{dp}{dx}\right]$$

$$0 = \left[p + 2x\frac{dp}{dx}\right] + \frac{p\left(p + 2x\frac{dp}{dx}\right)}{1+x^2p^4}$$

$\Rightarrow$ $\left[p + 2x\frac{dp}{dx}\right]\left[1 + \frac{p}{1+x^2p^4}\right] = 0$

Discarding $\left(1 + \frac{p}{1+x^2p^4}\right)$ as it gives singular solution. ...(2)

Therefore, eqn. (2) reduces to $p + 2x\frac{dp}{dx} = 0$

$\Rightarrow$ $2\frac{dp}{p} + \frac{dx}{x} = 0$

On integrating, we get

$2\log p + \log x = \log c$

$\Rightarrow$ $xp^2 = c$

$\Rightarrow$ $p = \sqrt{\frac{c}{x}}$

Therefore from eqn. (1), $y = 2\sqrt{\frac{c}{x}}\cdot x + \tan^{-1}(c)$

$\Rightarrow$ $y = 2\sqrt{cx} + \tan^{-1}(c)$.

Q 48. Solve $x\frac{dy}{dx} + y = x^3y^6$. (PTU, May 2011)

Solution. Dividing throughout the given equation by y^6, we have

$\frac{1}{y^6}\frac{dy}{dx} + \frac{1}{xy^5} = x^2$; which is of Leibnitz's form ...(1)

put $\frac{1}{y^5} = t \Rightarrow \frac{-5}{y^6}\frac{dy}{dx} = \frac{dt}{dx}$

Therefore, eqn. (1) gives ;

$$\frac{-1}{5}\frac{dt}{dx} + \frac{1}{x}\cdot t = x^2$$

$\Rightarrow$ $\frac{dt}{dx} - \frac{5}{x}t = -5x^2$ which is linear differential equation.

$$p = \frac{dy}{dx} = \frac{x}{y}\frac{dY}{dX} = \frac{\sqrt{X}}{\sqrt{Y}}P, \text{ Where } P = \frac{dY}{dX}$$

The given equation becomes,

$$\left(\frac{\sqrt{X}}{\sqrt{Y}}P.\sqrt{X} - \sqrt{Y}\right)\left(\frac{\sqrt{X}}{\sqrt{Y}}P.\sqrt{Y} + \sqrt{X}\right) = \frac{2\sqrt{X}}{\sqrt{Y}}P$$

$$(PX - Y)(P + 1) = 2P$$

$$PX - Y = \frac{2P}{P+1}$$

$$Y = PX - \frac{2P}{P+1} \text{ which is of Clairaut's form}$$

∴ Its solution is $Y = CX - \frac{2C}{C+1}$

and hence $y^2 = Cx^2 - \frac{2C}{C+1}$

Q 42. Define order and degree of an ordinary differential equation. **(PTU, Dec. 200[illegible])**

Solution. Order of an ordinary differential equation : The order of a differential equation is the order of the highest order derivative occuring in the differential equation.

The degree of differential equation is the degree of the highest order derivative which occurs, in the differential equation provided the equation has been made free of the radicals and fractions as far as the derivatives are concerned.

Q 43. State necessary conditions for an ordinary different equation to be exact. **(PTU, Dec. 200[illegible])**

Solution. The necessary condition for the differential equation

$Mdx + Ndy = 0$ to be exact is $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

Q 44. Prove that the necessary and sufficient condition for the differential equation $Mdx + Ndy = 0$, to be exact is $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. **(PTU, Dec. 200[illegible])**

Solution. (i) Necessary Condition :

Assume $Mdx + Ndy = 0$ is exact.

∴ $Mdx + Ndy = du$, where us is function of x and y.

But $du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$

∴ $Mdx + Ndy = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$

equating coeffs. of dx on both sides, $M = \frac{\partial u}{\partial x}$

equating coeff. of dy on both sides, $N = \frac{\partial u}{\partial y}$

$$\frac{\partial M}{\partial y} = \frac{\partial^2 u}{\partial y \partial x}, \frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x \partial y}$$

But $\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y}$ [∵ $\frac{\partial^2 u}{\partial x \partial y}$ and $\frac{\partial^2 u}{\partial y \partial x}$ are given to be continuous]

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Which is the required necessary condition.

(ii) Condition is sufficient :

Assume that $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

We have to prove that $Mdx + Ndy = 0$ is exact.

Let $\int M dx = u$ ...(1)

Where integration is performed on the supposition that y is const

$$\frac{\partial}{\partial x}\left[\int M dx\right] = \frac{\partial u}{\partial x} \text{ or } M = \frac{\partial u}{\partial x} \quad ...(2)$$

Also $\frac{\partial M}{\partial y} = \frac{\partial^2 u}{\partial y \partial x}$ ...(3)

But $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ (given) and $\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y}$

∴ from (3), $\frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x \partial y}$

or $\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)$

Integrating both sides w.r.t. 'x' regarding y as constant.

$$N = \frac{\partial u}{\partial y} + \text{a function of y}$$

or $N = \frac{\partial u}{\partial y} + f(y)$ say ...(4)

from (2) and (4), we get

$$Mdx + Ndy = \frac{\partial u}{\partial x}dx + \left[\frac{\partial u}{\partial y} + f(y)\right]dy$$

and solution is given by

$$y\,e^{\int P dx} = \int Q\,e^{\int P dx}dx + C$$

$$\Rightarrow \quad y \sin x = \int x \operatorname{cosec} x \sin x\, dx + C = \frac{x^2}{2} + C$$

Q 60. Determine whether the differential equation is exact $(x^2 + y^2 + 2x)\,dx + 2y\,dy = 0$. (PTU, Dec. 2020)

Ans. Here,

$$M = x^2 + y^2 + 2x\,;\; N = 2y$$

$$\frac{\partial M}{\partial y} = 2y\,;\; \frac{\partial N}{\partial x} = 0$$

$$\Rightarrow \quad \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

Therefore it is not an exact differential equation.

Now

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{2y - 0}{2y} = 1 \text{ which is a function of x only}$$

$$\text{I.F.} = e^{\int 1\,dx} = e^x$$

Multiply given equation by e^x, we get

$$e^x(x^2 + y^2 + 2x)\,dx + e^x\,2y\,dy = 0$$

which is exact and solution is given by

$$\int e^x (x^2 + y^2 + 2x)\,dx + \int 0\,dy = c$$

$$\Rightarrow \quad \int e^x x^2\,dx + y^2 e^x + \int 2xe^x\,dx = c$$

$$x^2 e^x - \int 2x \cdot e^x + y^2 e^x + \int 2x\,e^x = c$$

$$\Rightarrow \quad (x^2 + y^2)\,e^x = c \text{ is the required solution.}$$

Q 61. Solve the differential equation $\frac{dy}{dx} + \frac{x}{1-x^2}y = x\sqrt{y}$. (PTU, Dec. 2020)

Ans. Given diff. eqn. be, $\frac{dy}{dx} + \frac{x}{1-x^2}y = x\sqrt{y}$

$$\Rightarrow \quad \frac{1}{\sqrt{y}}\frac{dy}{dx} + \left(\frac{x}{1-x^2}\right)\sqrt{y} = x \quad \text{...(1)}$$

put

$$\sqrt{y} = t \Rightarrow \frac{1}{2\sqrt{y}}\frac{dy}{dx} = \frac{dt}{dx}$$

eqn (1) becomes,

$$2\frac{dt}{dx} + \frac{x}{1-x^2}t = x$$

$$\Rightarrow \quad \frac{dt}{dx} + \frac{xt}{2(1-x^2)} = \frac{x}{2}$$

Which is L.D.E in & and is of the form

$$\frac{dt}{dx} + Pt = Q$$

where $P = \frac{x}{2(1-x^2)}$; $Q = \frac{x}{2}$

$$\text{I.F.} = e^{\int P dx} = e^{\int \frac{x}{2(1-x^2)}dx} = e^{-\frac{1}{4}\int \frac{-2x\,dx}{1-x^2}}$$

$$= e^{-\frac{1}{4}\log(1-x^2)} = e^{\log(1-x^2)^{-1/4}} = (1-x^2)^{-1/4}$$

and solution be given by

$$t\,e^{\int P dx} = \int Q\,e^{\int P dx}dx + C$$

$$\Rightarrow \quad \sqrt{y}\,(1-x^2)^{-1/4} = \int \frac{x}{2}(1-x^2)^{-1/4}dx + C$$

$$= -\frac{1}{4}\int (1-x^2)^{-1/4}(-2x\,dx) + C$$

$$= -\frac{1}{4}\frac{(1-x^2)^{-\frac{1}{4}+1}}{-\frac{1}{4}+1} + C \qquad \left[\because \int [f(x)]^n f'(x)\,dx = \frac{[f(x)]^{n+1}}{n+1} + C\right]$$

$$= -\frac{1}{3}(1-x^2)^{3/4} + C$$

$$\Rightarrow \quad \sqrt{y} = -\frac{1}{3}(1-x^2) + C(1-x^2)^{1/4}$$

Q 62. Solve the differential $xyp^2 - (x^2 - y^2)\,p + xy = 0$, where $p = \frac{dy}{dx}$. (PTU, Dec. 2020)

Ans. The given diff. equation be

$$xyp^2 - (x^2 + y^2)\,p + xy = 0,\; p = \frac{dy}{dx}$$

$$\Rightarrow \quad xp\,(py - x) - y\,(py - x) = 0$$

$$\Rightarrow \quad (xp - y)\,(py - x) = 0$$

So its component equations are

$$xp - y = 0 \quad \text{...(1)}$$

$$py - x = 0 \quad \text{...(2)}$$

$$\Rightarrow \quad x\frac{dy}{dx} = y \quad \text{and} \quad y\frac{dy}{dx} = x$$

$$\Rightarrow \quad \frac{dy}{y} = \frac{dx}{x} \quad \text{and} \quad y\,dy = x\,dx$$

On integrating, we get

i.e. $\log y = \log x + \log c$ and $\frac{y^2}{2} = \frac{x^2}{2} + \frac{c}{2}$

$y = cx$ and $y^2 = x^2 + c$

Therefore, general solution of given equation is $(y - cx)(y^2 - x^2 - c) = 0$ ...(3)

❑❑❑

Multiply given eqn by xy^3, we have
$xy^3(2y^2+4x^2y)\,dx + xy^3(4xy+3x^2)\,dy = 0$
Which is exact and solution is given by

$$\int_{y=const} xy^2(2y^2+4x^2y)\,dx = C$$

$$2y^4\frac{x^2}{2} + 4y^3\frac{x^4}{4} = C$$

i.e. $x^2y^4 + y^3x^4 = C$ is the required solution.

Q 54. Solve $(x^2+y^2)(1+p)^2 = 2(x+y)(1+p)(x+yp) - (x+yp)^2$. (PTU, May 2011)

Solution. The given equation can be written as

$$x^2+y^2-2(x+y)\left(\frac{x+py}{1+p}\right)+\left(\frac{x+py}{1+p}\right)^2=0 \quad \text{...(1)}$$

put $x^2+y^2=Y,\ x+y=X$

$$2x+2yp=\frac{dY}{dx},\ 1+p=\frac{dX}{dx} \Rightarrow \frac{dY}{dX}=\frac{2(x+py)}{1+p}$$

$$P=\frac{2(x+py)}{1+p}, \text{ where } P=\frac{dY}{dX}$$

Therefore, eq. (1) becomes ; $Y-2X\frac{P}{2}+\left(\frac{P}{2}\right)^2=0 \Rightarrow Y=PX-\frac{P^2}{4}$

which is of Clairaut's form, its solution can be found out by replacing P by constant C

$$Y=cX-\frac{c^2}{4} \Rightarrow x^2+y^2=c(x+y)-\frac{c^2}{4} \text{ is the required solution.}$$

Q 55. Solve $x^2\left(\frac{dy}{dx}\right)^2+3xy\frac{dy}{dx}+2y^2=0$. (PTU, Dec. 2011)

Solution. The given equation is $x^2p^2+3xy\,p+2y^2=0$
$x^2p^2+2xyp+xyp+2y^2=0$
$\Rightarrow xp(px+2y)+y(px+2y)=0$
$\Rightarrow (px+y)(px+2y)=0$
Its component equations are
$px+y=0$ ...(1)
$px+2y=0$ ...(2)
From equations (1),

$$x\frac{dy}{dx}+y=0$$

$$\Rightarrow \frac{dy}{y}+\frac{dx}{x}=0$$

On integrating, we get
$\log y+\log x=\log c$
$\Rightarrow xy=c$
From equations (2),

$$x\frac{dy}{dx}+2y=0 \Rightarrow \frac{dy}{y}+2\frac{dx}{x}=0$$

On integrating, we get
$\log y+2\log x=\log c \Rightarrow x^2y=c$
Therefore, general solution of given equation is $(xy-c)(x^2y-c)=0$

Q 56. Form the differential equation from $y=e^x(A\cos x+B\sin x)$. (PTU, May 2002)

Ans. $y=e^x(A\cos x+B\sin x)$ ...(1) where A, B are arbitary constants
$y_1=e^x(-A\sin x+B\cos x)+(A\cos x+B\sin x)e^x$
$y_2=e^x(-A\cos x-B\sin x)+(-A\sin x+B\cos x)e^x+y_1$
$y_2=-y+(y_1-y)+y_1$ (using (1) and (2))
$y_2-2y_1+2y=0$ is the required diff. eq.

$$\Rightarrow \frac{d^2y}{dx^2}-2\frac{dy}{dx}+2y=0$$

Q 57. Solve the equation $y\,dx-x\,dy+3x^2y^2e^{x^3}dx=0$. (PTU, Dec. 2003)

Ans. The given diff. eq. can be written as

$$\frac{ydx-xdy}{y^2}+3x^2e^{x^3}dx=0$$

$$\Rightarrow d\left(\frac{x}{y}\right)+d\left(e^{x^3}\right)=0$$

on integrating, we get

$$\Rightarrow \frac{x}{y}+e^{x^3}=C \text{ is the req. sol.}$$

Q 58. Solve $xy\frac{dy}{dx}=1+x+y+xy$. (PTU, Dec. 2005)

Ans. $xy\frac{dy}{dx}=1+x+y(1+x)=(1+x)(1+y)$

$\Rightarrow \frac{y}{1+y}dy=\frac{(1+x)}{x}dx$, on integrating we get

$\Rightarrow y-\log|1+y|=\log|x|+x+c$

Q 59. Solve the differential equation $\frac{dy}{dx}+y\cot x=x\,\mathrm{cosec}\,x$. (PTU, Dec. 2008)

Ans. Given diff eqn. be, $\frac{dy}{dx}+y\cot x=x\,\mathrm{cosec}\,x$

which is of the form by $\frac{dy}{dx}+Py=Q$

where $P=\cot x$; $Q=x\,\mathrm{cosec}\,x$

Here $I.F.=e^{\int P dx}=e^{\int \cot x\,dx}=e^{\log \sin x}=\sin x$

$xp^2 - yp - y = 0$ ...(1)

i.e. $x = \frac{y(p+1)}{p^2}$; Diff w.r.t. y, we get

$$\frac{1}{p} = \frac{p+1}{p^2} + y\left[\frac{p^2\frac{dp}{dy} - (p+1)2p\frac{dp}{dy}}{p^4}\right]$$

i.e. $\frac{dp}{dy} = -\frac{p}{y(p+2)} \Rightarrow \frac{(p+2)}{p}dp = -\frac{dy}{y}$

$\Rightarrow$ $p + 2\log p = -\log y + C$

$\Rightarrow$ $y = A e^{-p-2\log p} = A e^{-p} p^{-2}$

Where, $A = e^{C}$ ...(2)

∴ eq (1) gives ;

$$x = \frac{(p+1)}{p^2} A e^{-p} p^{-2} = A e^{-p}(p+1)$$ ...(3)

Therefore eqn (2) and (3) gives the complete solution of eqn (1).

Q 51. Solve $\frac{dy}{dx} + \frac{y\cos x + \sin y + y}{\sin x + x\cos y + x} = 0.$ **(PTU, May 2011)**

Solution. The given differential equation can be written as,

$(\sin x + x\cos y + x)\,dy + (y\cos x + \sin y + y)\,dx = 0$

On comparing with $Mdx + Ndy = 0$

Here, $M = y\cos x + \sin y + y$; $N = \sin x + x\cos y + x$

$$\frac{\partial M}{\partial y} = \cos x + \cos y + 1 ; \frac{\partial N}{\partial x} = \cos x + \cos y + 1$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Therefore the given equation is exact and its solution is

$\int_{y\ const} (y\cos x + \sin y + y)\,dx + \int 0\,dy = 0 \Rightarrow y\sin x + x\sin y + xy = c$ is the required solution.

Q 52. Solve $y - 2px = \tan^{-1}(xp^2)$. **(PTU, May 2011)**

Solution. Given $y = 2px + \tan^{-1} xp^2$ ...(1)

Differentiate eqn. (1) w.r.t. x, we get

$$p = 2\left[p + x\frac{dp}{dx}\right] + \frac{1}{1+x^2p^4}\left[p^2 + 2xp\frac{dp}{dx}\right]$$

$$0 = \left[p + x\frac{dp}{dx}\right] + \frac{p\left(p + 2x\frac{dp}{dx}\right)}{1+x^2p^4}$$

$\Rightarrow \left[p + x\frac{dp}{dx}\right]\left[1 + \frac{p}{1+x^2p^4}\right] = 0$ ...(3)

Discarding $\left[1 + \frac{p}{1+x^2p^4}\right]$ gives singular solution

Therefore, eqn. (2) reduces to, $p + 2x\frac{dp}{dx} = 0$

$\Rightarrow$ $2\frac{dp}{p} + \frac{dx}{x} = 0$

On integrating, we get

$2\log p + \log x = \log c$

$\Rightarrow$ $xp^2 = c$

$\Rightarrow$ $p = \sqrt{\frac{c}{x}}$

Therefore from eqn. (1),

$$y = 2\sqrt{\frac{c}{x}}\,x + \tan^{-1}(c)$$

$\Rightarrow$ $y = 2\sqrt{cx} + \tan^{-1} c$.

Q 53. Solve $(2y^2 + 4x^2y)\,dx + (4xy + 3x^3)\,dy = 0.$ **(PTU, May 2011)**

Solution. Compare the given diff. eqn with $Mdx + Ndy = 0$

Here, $M = 2y^2 + 4x^2y$; $N = 4xy + 3x^3$

$$\frac{\partial M}{\partial y} = 4y + 4x^2 ; \frac{\partial N}{\partial x} = 4y + 9x^2$$

Here $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ given diff. eqn is not exact.

The given diff. eqn can be written as

$(2y^2\,dx + 4xy\,dy) + (4x^2y\,dx + 3x^3\,dy) = 0$

$\Rightarrow y(2y\,dx + 4x\,dy) + x^2(4y\,dx + 3x\,dy) = 0$

Comparing with $x^a y^b(my\,dx + nx\,dy) + x^{a'} y^{b'}(m'y\,dx + n'x\,dy) = 0$

Here $a = 0, b = 1, m = 2, n = 4$

$a' = 2, b' = 0, m' = 4, n' = 3$

Where $\frac{a+h+1}{m} = \frac{b+K+1}{n}$ i.e. $\frac{h+1}{2} = \frac{K+2}{4} \Rightarrow 2h - K = 0$

and $\frac{a'+h+1}{m'} = \frac{b'+K+1}{n'}$ i.e. $\frac{h+3}{4} = \frac{K+1}{3} \Rightarrow 3h - 4K = -5$

on solving $h = 1$; $K = 2$

$IF = x^h y^k = xy^2$

LEGENDRE'S LINEAR EQ :

If the eq. is of the form

$$a_0(a+bx)^n \frac{d^n y}{dx^n} + a_1(a+bx)^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \dots + a_n y = Q$$

Here we put $\quad a+bx = e^z \Rightarrow \log(a+bx) = z$

and $\quad \theta = \frac{d}{dz}$

$(a+bx)D = b\theta,\ (a+bx)^2D^2 = b^2\theta(\theta-1)$ and so on

QUESTION-ANSWERS

Q 1. Solve : $\frac{dx}{dt} = -2x + y$

$\frac{dy}{dt} = -4x + 3y + 10\cos t.$ (PTU, Dec. 200)

Ans. The given diff. eqs can be written as

$(D+2)x - y = 0$ ...(1)

and $\quad 4x + (D-3)y = 10\cos t$ ...(2)

Multiply eq (1) by $(D-3)$ + eq (2), we get

$(D+2)(D-3)x + 4x = 10\cos t \Rightarrow [D^2 - D - 2]x = 10\cos t$

Its A.E. be $D^2 - D - 2 = 0 \Rightarrow D = 2, -1$

i.e. $\quad C.F. = C_1e^{-t} + C_2e^{2t}$

$$P.I. = 10\frac{1}{D^2-D-2}\cos t = 10\frac{1}{-1-D-2}\cos t = 10\frac{(-3+D)}{9-D^2}\cos t$$

$$= \frac{10}{10}[-3\cos t - \sin t]$$

$$x = C_1e^{-t} + C_2e^{2t} - 3\cos t - \sin t$$

$$\frac{dx}{dt} = -C_1e^{-t} + 2C_2e^{2t} + 3\sin t - \cos t$$

eq (1) gives

$$y = -C_1e^{-t} + 2C_2e^{2t} + 3\sin t - \cos t + 2C_1e^{-t} + 2C_2e^{2t} - 6\cos t - 2\sin t$$
$$= C_1e^{-t} + 4C_2e^{2t} + \sin t - 7\cos t$$

Q 2. Write the particular integral of (PTU, May 200)
$(D^2 - 2D + 4)y = e^x \sin x$.

Ans. $P.I. = \frac{1}{D^2-2D+4}e^x\sin x = e^x\frac{1}{(D+1)^2-2(D+1)+4}\sin x$

$$= e^x\frac{1}{D^2+3}\sin x = \frac{e^x \sin x}{2}$$

Q 3. Find the particular solution of the differential equation

$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = e^{e^x}$. (PTU, Dec. 2003)

Ans. $P.I. = \frac{1}{D^2+3D+2}e^{e^x} = \frac{1}{(D+1)(D+2)}e^{e^x}$

$$= \left[\frac{1}{D+1} - \frac{1}{D+2}\right]e^{e^x}$$

$$= e^{-x}\int e^x e^{e^x}dx - e^{-2x}\int e^{2x}e^{e^x}dx \quad \left[\because \frac{1}{D-\alpha}X = e^{\alpha x}\int Xe^{-\alpha x}dx\right]$$

$$= e^{-x}\int e^t dt - e^{-2x}\int e^t t\, dt$$

$$= e^{-x}e^{e^x} - e^{-2x}e^{e^x}(e^x - 1) = e^{-2x}e^{e^x}$$

Q 4. Write the particular integral of, $(D^2 - 3D + 2)y = 2e^x\cos\frac{x}{2}$. (PTU, Dec. 2005)

Ans. $P.I. = \frac{1}{D^2-3D+2}2e^x\cos\frac{x}{2} = 2e^x\frac{1}{(D+1)^2-3(D+1)+2}\cos\frac{x}{2}$

$$= 2e^x\frac{1}{D^2-D}\cos\frac{x}{2} = 2e^x\frac{1}{-\frac{1}{4}-D}\cos\frac{x}{2}$$

$$= -2e^x\frac{\left(D-\frac{1}{4}\right)}{D^2-\frac{1}{16}}\cos\frac{x}{2} = -2e^x\frac{\left[-\frac{1}{2}\sin\frac{x}{2} - \frac{1}{4}\cos\frac{x}{2}\right]}{-\frac{1}{4}-\frac{1}{16}}$$

$$= \frac{32}{5}e^x\left[\frac{-1}{2}\sin\frac{x}{2} - \frac{1}{4}\cos\frac{x}{2}\right]$$

Q 5. Explain briefly the method of variation of parameters to find the particular solution of a differential equation. (PTU, May 2004)

Ans. This method is applicable to diff. eqs of the form $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = X$

then $y = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)$

Case IV : If this Non real root repeated 2 times then

$$y = e^{\alpha x}[(c_1 + c_2 x)\cos \beta x + (c_3 + c_4 x)\sin \beta x]$$

Five rules for finding particular integral :

Rule-I $\frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax}$ if $f(a) \neq 0$ (i.e. put $D = a$)

Case of failure if $f(a) = 0$

$$\frac{1}{f(D)} e^{ax} = x \frac{1}{\frac{d}{dD}[f(D)]} e^{ax}, \text{ Then replace } D = a$$

Rule-II $\frac{1}{f(D^2)} \cos ax = \frac{1}{f(-a^2)} \cos ax$ similar formula for sin ax

if $f(-a^2) \neq 0$

Case of failure : if $f(-a^2) = 0$

Then $\frac{1}{f(D^2)} \cos ax = x \frac{1}{\frac{d}{dD}[f(D^2)]} \cos ax$ then replace $D^2 = -a^2$

Rule-III $\frac{1}{f(D)} x^m$ expand $[f(D)]^{-1}$ by binomial theorem so far as the term D^m

Then operate x^m term by term.

Rule-IV $\frac{1}{f(D)} e^{ax} \cdot V$ where V is a function of x

$$\frac{1}{f(D)} e^{ax} \cdot V = e^{ax} \frac{1}{f(D+a)} \cdot V$$

Rule-V $\frac{1}{f(D)} x \cdot V$ where V is a function of x

$$\frac{1}{f(D)} x \cdot V = x \frac{1}{f(D)} V + \frac{d}{dD}\left[\left(\frac{1}{f(D)}\right)\right] V$$

or

$$x \frac{1}{f(D)} V - \frac{1}{f(D)}\left\{f'(D)\left(\frac{1}{f(D)}(V)\right)\right\}$$

Note : $\frac{1}{(D-a)} Q = e^{ax} \int e^{-ax} Q\,dx$

also $\frac{1}{D} Q = \int Q\,dx$

Another two methods for finding particular integrals :

Ist : Variation of parameter : This method is applied to equations of the form

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = X$$

Where P, Q are constants and X is a function of x only

So its P.I $= u y_1 + v y_2$

where $u = -\int \frac{y_2 X}{W} dx$, $v = \int \frac{y_1 X}{W} dx$

where W = Wronskian of y_1 and $y_2 = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

and y_1, y_2 are solutions of

$$y'' + Py' + Qy = 0$$

IInd : Method of undetermined coefficients :

We can also find the P.I of $f(D)\,y = X$ by inspection. Here trial solution is totally depends on the form of the function X

i.e. when (i) $X = 3e^{-x}$, trial solution $= ae^{-x}$

(ii) $X = 3 \sin x$, trial solution $= c_1 \sin x + c_2 \cos x$

(iii) $X = 3x^2$, trial solution $= c_1 x^2 + c_2 x + c_3$

Now If $X = \tan x$ or $\sec x$ then method fails.

Note : If the trial sol. appears in C.F or any term of trial sol. present in C.F then we multiply the trial sol. by lowest positive integral power of x so that No term of the trial solution appears in C.F.

CAUCHY'S LINEAR EQUATION

If the eq. is of the form

$$P_0 x^n \frac{d^n y}{dx^n} + P_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \dots + P_n y = Q$$

where $P_0, \dots P_n$ are constants

Here we put $x = e^z \Rightarrow \log x = z, x > 0$

and $xD = \theta$ where $\theta = \frac{d}{dz}$

$x^2D^2 = \theta(\theta - 1)$

$x^3 D^3 = \theta(\theta - 1)(\theta - 2)$ and so on

FOR NOTES

Module 4

Syllabus

Second and higher order linear differential equations with constant coefficients, method of variation of parameters, Equations reducible to linear equations with constant coefficients Cauchy and Legendre's equations.

BASIC CONCEPTS

Linear Diff. eqs. with Constant-Coefficients :

The general Linear differential eq. of order n is $P_0 \frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} \ldots + P_n y = Q$

Where $P_0, P_1 \ldots P_n$, Q are functions are functions of x or constants

i.e. $(P_0 D^n + P_1 D^{n-1} \ldots + P_n) y = Q$

where $D = \frac{d}{dx}, D^2 = \frac{d^2}{dx^2} \ldots D^n = \frac{d^n}{dx^n}$

If $P_0, P_1 \ldots P_n$ are all constants and Q is a function of x then it is a linear diff. eq. of order n with constant coefficients.

If $Q = 0$ then it is Homogeneous L.D. eq.

If $Q \neq 0$ then it is represented by $[f(D)] y = Q$

Auxiliary eq. of $[f(D)] y = 0$ ... (1) can be obtained by replacing D by m i.e. $f(m) = 0$

i.e. $P_0 m^n + P_1 m^{n-1} \ldots + P_n = 0$ gives A.E.

It is nth degree eq. (with real coefficients), so it has exactly n-roots say $m_1, m_2, m_3 \ldots m_n$

Four cases arises :

Case I : If all the roots $m_1, m_2 \ldots m_n$ are real and distinct

Then general sol. of (1) is given by

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} \ldots + c_n e^{m_n x}$$

where $c_1, c_2 \ldots c_n$ are arbitrary constants

Case II : If m_1 repeated r_1 times, m_2 repeated r_2 times similarly m_k repeated r_k times s.t. $r_1 + r_2 + r_3 + \ldots + r_k = n$

sol. of (1) is given by

$$y = (c_1 + c_2 x + c_3 x^2 \ldots c_{r_1} x^{r_1 - 1}) e^{m_1 x} + \ldots (c_1 + c_2 x \ldots + c_{r_k} x^{r_k - 1}) e^{m_k x}$$

Case III : Suppose eq. (1) has non-real (complex) root say $\alpha \pm i\beta$

$$= e^{-x}\frac{1}{D^2+1}\sec x = e^{-x}\frac{1}{(D-i)(D+i)}\sec x$$

$$= e^{-x}\left[\frac{\frac{1}{2i}}{D-i} - \frac{\frac{1}{2i}}{D+i}\right]\sec x$$

$$= e^{-x}\cdot\frac{1}{2i}\left[\frac{1}{D-i}\sec x - \frac{1}{D+i}\sec x\right] \quad \ldots(1)$$

$$\frac{1}{D-i}\sec x = e^{ix}\int e^{-ix}\sec x\,dx = e^{ix}\int(\cos x - i\sin x)\sec x\,dx$$

$$= e^{ix}\left[\int dx - i\int\frac{\sin x}{\cos x}dx\right]$$

$$= e^{ix}(x + i\log|\cos x|)$$

$$\frac{1}{D+i}\sec x = e^{-ix}(x - i\log|\cos x|)$$

∴ eq (1) gives

$$= e^{-x}\cdot\frac{1}{2i}\left[x\left(e^{ix} - e^{-ix}\right) + i\log|\cos x|\left(e^{ix} + e^{-ix}\right)\right]$$

$$\text{P.I.} = e^{-x}[x\sin x + \cos x\log|\cos x|]$$

$$y = \text{C.F.} + \text{P.I.}$$

Q 19. Solve the differential equation

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = xe^x\sin x.$$ **(PTU, Dec. 2010, 2009, 2003)**

Ans. The given diff. eq. can be written as $(D^2 - 2D + 1)y = xe^x\sin x$

Its Auxiliary eq. be $D^2 - 2D + 1 \Rightarrow D = 1, 1$

$$\text{C.F.} = (C_1 + C_2x)e^x$$

$$\text{P.I.} = \frac{1}{D^2 - 2D + 1}xe^x\sin x = \frac{1}{(D-1)^2}e^x(x\sin x)$$

$$= e^x\cdot\frac{1}{[D+1-1]^2}x\sin x = e^x\cdot\frac{1}{D^2}x\sin x$$

$$= e^x\left[x\cdot\frac{1}{D^2}\sin x - \frac{1}{D^2}\left\{2D\left(\frac{1}{D^2}\sin x\right)\right\}\right]$$

$$= e^x\left[-x\sin x + \frac{1}{D^2}(2D\sin x)\right]$$

$$= e^x(-x\sin x - 2\cos x)$$

$$y = \text{C.F.} + \text{P.I.}$$

$$= (C_1 + C_2x)e^x + e^x(-x\sin x - 2\cos x)$$

Q 20. Solve by the method of variation of parameters:

$$\frac{d^2y}{dx^2} + y = \sec x$$ **(PTU, Dec. 2020 ; May 2005)**

Ans. The given differential eq. be $\frac{d^2y}{dx^2} + y = \sec x$

Its symbolic form be $(D^2+1)y = \sec x$

Its A.E is $D^2+1 = 0 \Rightarrow D = \pm i$

Complementary function = C.F = $C_1\cos x + C_2\sin x$

Let the particular integral = P.I = $uy_1 + vy_2$

where $y_1 = \cos x$; $y_2 = \sin x$; $X = \sec x$

Now $$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$$

Now $$u = -\int\frac{y_2X}{W}dx = -\int\frac{\sin x.\sec x}{1}dx = \text{Log}|\cos x|$$

$$v = \int\frac{y_1X}{W}dx = \int\cos x.\sec x\,dx = x$$

$$\text{P.I} = \cos x\,\text{Log}|\cos x| + x\sin x$$

$$y = \text{C.F} + \text{P.I} = C_1\cos x + C_2\sin x + \cos x\,\text{Log}|\cos x| + x\sin x$$

Q 21. $(D^2 - 6D + 13)y = 8e^{3x}\sin 4x + 2^x$. **(PTU, Dec. 2005)**

Ans. Its A.E. is given by $D^2 - 6D + 13 = 0$

$$D = \frac{6 \pm \sqrt{36-52}}{2} = \frac{6 \pm 4i}{2} = 3 \pm 2i$$

$$\text{C.F.} = (C_1\cos 2x + C_2\sin 2x)e^{3x}$$

and $$\text{P.I.} = \frac{1}{D^2 - 6D + 13}8e^{3x}\sin 4x + 2^x$$

$$= 8\cdot\frac{1}{D^2 - 6D + 13}\left(e^{3x}\sin 4x\right) + \frac{1}{D^2 - 6D + 13}2^x$$

$$= 8e^{3x}\frac{1}{(D+3)^2 - 6(D+3) + 13}\sin 4x + \frac{1}{D^2 - 6D + 13}e^{\log 2^x}$$

$$(a+bx)^n \frac{d^n y}{dx^n} + (a+bx)^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \ldots + (a+bx)\frac{dy}{dx} + y = X$$

where X be any function of x.

The given diff. eq. can be written as

$$[(a+bx)^n D^n + (a+bx)^{n-1} D^{n-1} + \ldots + (a+bx)D + 1]\, y = X \qquad \text{---(1)}$$

put $a + bx = e^t \Rightarrow \text{Log}(a+bx) = t$

$(a+bx)D = b\theta,\ (a+bx)^2 D^2 = b^2\theta(\theta - 1)$ and so on

where $\theta = \frac{d}{dt}$

eq (1) reduces to L.D.E with constant coefficients.

Q 13. Find the particular integral of the equation $4y'' - 4y' + y = e^{x/2}$ (PTU, Dec. 2007)

Solution.

$$\text{P.I.} = \frac{1}{4D^2 - 4D + 1} e^{x/2} \text{ where } D = \frac{d}{dx}$$

$$= \frac{1}{4\left(\frac{1}{2}\right)^2 - 4\cdot\frac{1}{2} + 1} e^{x/2} \text{ [Case of failure]}$$

$$= x \cdot \frac{1}{8D - 4} e^{x/2} \text{ [Case of failure]}$$

$$= x^2 \cdot \frac{1}{8} e^{x/2}$$

Q 14. Find the complementary function of the equation $y'' + 4y' + 3y = x \sin 2x$. (PTU, Dec. 2007)

Solution. The given diff. eq. can be written as

$(D^2 + 4D + 3)y = x \sin 2x$ where $D = \frac{d}{dx}$

A.E is given by $D^2 + 4D + 3 = 0$

$D = -1, -3$

∴ Complementary function $C.F = C_1 e^{-x} + C_2 e^{-3x}$

Q 15. Find complementary solution of $9y''' + 3y'' - 5y' + y = 0$. (PTU, May 2008)

Solution. The given eq. can be written as

$(9D^3 + 3D^2 - 5D + 1)y = 0$

Its auxiliary eq. be $9m^3 + 3m^2 - 5m + 1 = 0$

⇒ $(m+1)(9m^2 - 6m + 1) = 0$

⇒ $(m+1)(3m-1)^2 = 0 \Rightarrow m = -1, \frac{1}{3}, \frac{1}{3}$

Complementary function or complete solution is given by

$$y = C_1 e^{-x} + (C_2 + C_3 x) e^{x/3}$$

Q 16. Find particular integral of $y''' - y'' + 4y' - 4y = \sin 3x$. (PTU, May 2008)

Solution. The given diff. eq. can be written as

$(D^3 - D^2 + 4D - 4)y = \sin 3x$

$$\text{P.I.} = \frac{1}{D^3 - D^2 + 4D - 4} (\sin 3x)$$

$$= \frac{1}{-9D + 9 + 4D - 4} (\sin 3x) \text{ [replacing } D^2 \text{ by } -9]$$

$$= \frac{1}{-5(D-1)} (\sin 3x)$$

$$= \frac{-1}{5} \frac{(D+1)}{D^2 - 1} (\sin 3x)$$

$$= \frac{1}{50} (3\cos 3x + \sin 3x)$$

Q 17. Find the particular integral of $\frac{d^3y}{dx^3} + 4\frac{dy}{dx} = \sin 2x$. (PTU, Dec. 2006)

Solution. The given diff. eq. can be written as

$(D^3 + 4D)y = \sin 2x$

$$\text{P.I.} = \frac{1}{D^3 + 4D} (\sin 2x) \text{ (by replacing } D^2 \text{ by } -2^2\text{, we get case of failure)}$$

$$= x \cdot \frac{1}{3D^2 + 4} (\sin 2x) \qquad \left[\frac{1}{F(D)} \sin ax = x \frac{1}{\frac{d}{dD}F(D)} \sin ax\right]$$

$$= x \cdot \frac{1}{-3(2)^2 + 4} \sin 2x$$

$$= \frac{x}{-8} \sin 2x$$

Q 18. Solve : $(D^2 + 2D + 2)y = e^{-x} \sec x$. (PTU, Dec. [illegible])

Ans. Its A.E. be $D^2 + 2D + 2 = 0 \Rightarrow D = \frac{-2 \pm \sqrt{4-8}}{2} = \frac{-2 \pm 2i}{2} = -1 \pm i$

$C.F. = (C_1 \cos x + C_2 \sin x) e^{-x}$

$$\text{P.I.} = \frac{1}{D^2 + 2D + 2} e^{-x} \sec x = e^{-x} \frac{1}{(D-1)^2 + 2(D-1) + 2} \sec x$$

Where P, Q are constants and X be a function of x only

Let y_1, y_2 be two solutions of $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0$

so its particular integral P.I. $= uy_1 + vy_2$

where $u = -\int \frac{y_2 X}{W} dx$; $v = \int \frac{y_1 X}{W} dx$

and $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$ = Wronskian of y_1 and y_2

Q 6. Write the most general Cauchy's homogeneous linear differential equation. (PTU, Dec. 2004)

Ans. A diff. eq. of the form

$$a_0 x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1}y}{dx^{n-1}} + \ldots + a_n y = Q$$

i.e. $(a_0 x^n D^n + a_1 x^{n-1} D^{n-1} \ldots + a_n) y = Q$

where $D = \frac{d}{dx}$, a_i's are constant and Q be any function of x.

Here we put $x = e^z \Rightarrow \log x = z$

s.t. $xD = \theta$, $x^2D^2 = \theta(\theta - 1)$ ..... $x^nD^n = \theta(\theta - 1)$ ..... $(\theta - \overline{n-1})$

We get linear differential eq. with constant coefficients.

Q 7. Define a linear differential equation. Also give an example of a linear differential equation. (PTU, May 2005)

Ans. A differential eq. is said to be linear if

(i) Every dependent variable and its derivative occurs in the diff. eq. are of first degree.

(ii) No product of dependent variable and its derivative occurs.

e.g. (a) $\frac{dy}{dx} = x^2 + 1$ (b) $\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 0$

Q 8. Write the most general Legendre's linear differential equation. (PTU, Dec. 2005)

Ans. Legendre's linear differential eq. is of the form

$$a_0 (a+bx)^n \frac{d^n y}{dx^n} + a_1 (a+bx)^{n-1} \frac{d^{n-1}y}{dx^{n-1}} + \ldots + a_n y = Q$$

Where a_i's are constant and Q is an function of x

For its solution we put $a + bx = e^z \Rightarrow \log(a+bx) = z$

s.t. $(a+bx)D = b\theta$, $(a+bx)^2 D^2 = b^2\theta(\theta - 1)$ and so on, $\theta = \frac{d}{dz}$

We get L.D.E with constant coefficients.

Q 9. Solve $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = e^x \sin x$ (PTU, Dec. 2011; May 2006)

Ans. The given diff. eq. can be written as $(D^2 - 2D + 1)y = e^x \sin x$

Its A.E. is $D^2 - 2D + 1 = 0 \Rightarrow D = 1, 1$

C.F. $= (C_1 + C_2 x)e^x$

P.I. $= \frac{1}{(D-1)^2} e^x \sin x = e^x \frac{1}{(D+1-1)^2} \sin x = e^x \frac{1}{D^2} \sin x$

$= -e^x \sin x$

$y = (C_1 + C_2 x)e^x - e^x \sin x$

Q 10. Solve $x^2 y'' + 4xy' + 2y = 0$ (PTU, May 2006)

Ans. The given diff. eq. is of the form $(x^2D^2 + 4xD + 2)y = 0$

which is of Cauchy's form

put $x = e^z \Rightarrow \log x = z$

i.e. $xD = \theta$, $x^2D^2 = \theta(\theta - 1)$ where $\theta = \frac{d}{dz}$

$\Rightarrow (\theta(\theta - 1) + 4\theta + 2)y = 0 \Rightarrow (\theta^2 + 3\theta + 2)y = 0$

Its A.E. is $\theta^2 + 3\theta + 2 = 0$

$\Rightarrow$ $\theta = -1, -2$

$y = C_1 e^{-z} + C_2 e^{-2z} = \frac{C_1}{x} + \frac{C_2}{x^2}$

Q 11. If $y_1 = \frac{1}{x}$ is a solution of the differential equation $x^2y'' + 4xy' + 2y = 0$. Find the second linearly independent solution and write the general solution. (PTU, May 2006)

Ans. The given diff. eq. can be written as $(x^2D^2 + 4xD + 2)y = 0$ ..... (1)

where $D = \frac{d}{dx}$

It is of Cauchy's form

put $x = e^z \Rightarrow \log x = z$, $xD = \theta$, $x^2D^2 = \theta(\theta - 1)$

where $\theta = \frac{d}{dz}$ eq (1) gives $[\theta(\theta - 1) + 4\theta + 2]y = 0$

i.e. $[\theta^2 + 3\theta + 2]y = 0$ Its A.E. be $\theta^2 + 3\theta + 2 = 0 \Rightarrow \theta = -1, -2$

$y = c_1 e^{-z} + c_2 e^{-2z} = \frac{C_1}{x} + \frac{C_2}{x^2}$ be the general solution

IInd L.I. solution be $= \frac{1}{x^2}$

Q 12. How Legendre's differential equation can be reduced to differential equation with constant co-efficients. Explain. (PTU, Dec. 2006)

Ans. The Legendre's form of diff. eq be

$= 8 \cos 2x$

$$v = \int \frac{y_1 X}{W} dx = \int \frac{\cos 4x \cdot 32 \sec 2x}{4} dx$$

$$= 8 \int \frac{\cos 4x}{\cos 2x} dx = 8 \int \frac{(2\cos^2 2x - 1)}{\cos 2x} dx$$

Thus $$v = 8\left[\sin 2x - \frac{\log|\sec 2x + \tan 2x|}{2}\right]$$

∴ From eq (1); we have

$$P.I. = 8 \cos 2x \cos 4x + \sin 4x \,[8 \sin 2x - 4 \log|\sec 2x + \tan 2x|]$$
$$= 8 \cos 2x - 4 \sin 4x \log|\sec 2x + \tan 2x|$$

Thus complete solution is given by

$$y = C_1 \cos 4x + C_2 \sin 4x + 8 \cos 2x - 4 \sin 4x \log|\sec 2x + \tan 2x|$$

Q 27. Solve $y'' - 2y' + y = e^x \log x$, using method of variation. (PTU, Dec. 20..)

Solution. The given diff. eq can be written as

$$(D^2 - 2D + 1)y = e^x \log x \,;\; D = \frac{d}{dx}$$

Its A.E. is $D^2 - 2D + 1 = 0 \Rightarrow D = 1, 1$

$$C.F. = (C_1 + C_2 x)e^x$$

and $$P.I. = uy_1 + vy_2 \quad \text{---(1)}$$

Here $y_1 = e^x$; $y_2 = xe^x$ and $X = e^x \log x$ ---(2)

Now $$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x & xe^x \\ e^x & e^x(x+1) \end{vmatrix}$$
$$= e^{2x}(x+1) - xe^{2x} = e^{2x}$$

Now $$u = -\int \frac{y_2 X}{W} dx = -\int \frac{xe^x \cdot e^x \log x \, dx}{e^{2x}} = -\left[\log x \cdot \frac{x^2}{2} - \frac{x^2}{4}\right]$$

and $$v = \int \frac{y_1 X}{W} dx = \int \frac{e^x \cdot e^x \log x}{e^{2x}} dx$$
$$= x \log x - x$$

eq (2) gives

$$P.I. = \left(-\frac{x^2}{2}\log x + \frac{x^2}{4}\right)e^x + (x \log x - x)\,xe^x$$

Complete solution is given by

$$y = (C_1 + C_2 x)e^x + \left(\frac{x^2}{2}\log x - \frac{3}{4}x^2\right)e^x$$

Q 28. Find the particular solution of the differential equation $y'' + a^2 y = \sec ax$. (PTU, May 2012)

Solution. The given diff. eqn can be written as

$(D^2 + a^2)y = \sec ax$

$$P.I. = \frac{1}{(D^2 + a^2)}(\sec ax) = \frac{1}{(D - ai)(D + ai)}(\sec ax)$$
$$= \left[\frac{-1/2ai}{D + ai} + \frac{1/2ai}{D - ai}\right](\sec ax)$$
$$= \frac{1}{2ai}\left[\frac{1}{D - ai}(\sec ax) - \frac{1}{D + ai}(\sec ax)\right] \quad (1)$$

$$\frac{1}{D - ai}(\sec ax) = e^{aix}\int e^{-aix} \sec ax \, dx \qquad \left[\because \frac{1}{D - \alpha}Q = e^{\alpha x}\int e^{-\alpha x} Q\,dx\right]$$
$$= e^{aix}\int (\cos ax - i \sin ax)\sec ax \, dx$$
$$= e^{aix}\left[x + \frac{i}{a}\log|\cos ax|\right] \quad (2)$$

$$\frac{1}{D + ai}(\sec ax) = e^{-aix}\left[x - \frac{i}{a}\log|\cos ax|\right] \quad (3)$$

putting eqn (2) and (3) in eqn (1); we have

i.e. $$P.I. = \frac{1}{2ai}\left[e^{aix}\left(x + \frac{i}{a}\log|\cos ax|\right) - e^{-aix}\left(x - \frac{i}{a}\log|\cos ax|\right)\right]$$
$$= \frac{1}{2ai}\left[x\left(e^{aix} - e^{-aix}\right) + \frac{i \log|\cos ax|}{a}\left(e^{aix} + e^{-aix}\right)\right]$$
$$= \frac{x}{a}\sin ax + \frac{\log|\cos ax|}{a^2}\cos ax$$

Q 29. Solve : $x^3\frac{d^3y}{dx^3} + 2x^2\frac{d^2y}{dx^2} + 2y = 10\left(x + \frac{1}{x}\right)$ (PTU, May 2003)

Ans. The given diff. eq. can be written as

$[x^3D^3 + 2x^2D^2 + 2]y = 10\left(x + \frac{1}{x}\right)$ where $D = \frac{d}{dx}$

It is of Cauchy's form

put $x = e^z \Rightarrow \log x = z$

s.t. $xD = \theta$, $x^2D^2 = \theta(\theta - 1)$, $x^3D^3 = \theta(\theta - 1)(\theta - 2)$, where $\theta = \frac{d}{dz}$

$[\theta(\theta - 1)(\theta - 2) + 2\theta(\theta - 1) + 2]y = 10(e^z + e^{-z})$

$(\theta^2(\theta - 1) + 2)y = 10(e^z + e^{-z})$

Its A.E. be $\theta^3 - \theta^2 + 2 = 0 \Rightarrow (\theta + 1)(\theta^2 - 2\theta + 2) = 0$

$$= \frac{1}{D^2+D+1}\left[1+\frac{1-\cos 2x}{2}+2\sin x\right]$$

$$= \frac{1}{D^2+D+1}\cdot\frac{3}{2} - \frac{1}{2}\frac{1}{D^2+D+1}\cos 2x + 2\frac{1}{D^2+D+1}\sin x$$

$$= \frac{3}{2}\underset{\text{(rule-I)}}{\frac{1}{D^2+D+1}e^{0x}} - \frac{1}{2}\underset{\text{(rule-II)}}{\frac{1}{D^2+D+1}\cos 2x} + 2\underset{\text{(rule-II)}}{\frac{1}{D^2+D+1}\sin x}$$

$$= \frac{3}{2}\cdot\frac{1}{1} - \frac{1}{2}\frac{1}{D-3}\cos 2x + 2\frac{1}{-1+D+1}\sin x$$

$$= \frac{3}{2} - \frac{1}{2}\frac{D+3}{D^2-9}(\cos 2x) + 2(-\cos x)$$

$$= \frac{3}{2} - \frac{1}{2}\frac{(D+3)}{-13}(\cos 2x) + 2(-\cos x)$$

$$= \frac{3}{2} + \frac{1}{26}\left[-2\sin 2x + 3\cos 2x\right] - 2\cos x$$

C.S $= y = $ C.F + P.I.

$$y = e^{-x/2}\left[c_1\cos\frac{\sqrt{3}}{2}x + c_2\sin\frac{\sqrt{3}}{2}x\right] + \frac{3}{2} - 2\cos x + \frac{1}{26}\left[-2\sin 2x + 3\cos 2x\right]$$

Q 25. Solve the system of equations :
$(2D-4)y_1 + (3D+5)y_2 = 3t+2,\ (D-2)y_1 + (D+1)y_2 = t$

(PTU, Dec. 2007)

Solution. The given diff. eqs are

$$(2D-4)y_1 + (3D+5)y_2 = 3t+2 \quad \text{...(1)}$$
$$(D-2)y_1 + (D+1)y_2 = t \quad \text{...(2)}$$

eq (1) − 2 × eq (2) ; we get

$$(3D+5-2D-2)y_2 = 3t+2-2t$$
$$\Rightarrow (D+3)y_2 = t+2$$

A.E. is given by $D+3=0 \Rightarrow D=-3$

$$\text{C.F} = C_1 e^{-3t}$$

$$\text{P.I} = \frac{1}{D+3}(t+2) = \frac{1}{3}\left[1+\frac{D}{3}\right]^{-1}(t+2)$$

$$= \frac{1}{3}\left[1-\frac{D}{3}\right](t+2) = \frac{1}{3}\left[t+2-\frac{1}{3}\right] = \frac{1}{3}\left[t+\frac{5}{3}\right]$$

$$y_2 = \text{C.F} + \text{P.I} = C_1 e^{-3t} + \frac{1}{3}\left(t+\frac{5}{3}\right)$$

again multiply eq (1) by $(D+1)$ and eq (2) by $(3D+5)$ and subtracting, we get

$$[(2D-4)(D+1) - (D-2)(3D+5)]y_1 = (D+1)(3t+2) - (3D+5)t$$
$$\Rightarrow (-D^2-D+6)y_1 = 3+3t+2-3-5t$$
$$\Rightarrow (D^2+D-6)y_1 = 2t-2$$

A.E. is given by $D^2+D-6=0 \Rightarrow (D-2)(D+3)=0$

$$D = 2, -3$$
$$\text{C.F} = C_3 e^{2t} + C_4 e^{-3t}$$

$$\text{P.I} = \frac{1}{D^2+D-6}(2t-2)$$

$$= \frac{1}{-6\left[1-\frac{D^2}{6}-\frac{D}{6}\right]}(2t-2)$$

$$= -\frac{1}{3}\left[1-\left(\frac{D^2}{6}+\frac{D}{6}\right)\right]^{-1}(t-1)$$

$$= -\frac{1}{3}\left[1+\left(\frac{D^2}{6}+\frac{D}{6}\right)\right](t-1)$$

$$= -\frac{1}{3}\left[t-1+\frac{1}{6}\right] = -\frac{1}{3}\left[t-\frac{5}{6}\right]$$

$$y_1 = C_3 e^{2t} + C_4 e^{-3t} - \frac{1}{3}\left[t-\frac{5}{6}\right]$$

Q 26. Find the general solution of the equation $y'' + 16y = 32\sec 2x$, using method of variation of parameters.

(PTU, May 2010, 2008)

Solution. The given diff. eq. be $y'' + 16y = 32\sec 2x$

Its symbolic form is $(D^2+16)y = 32\sec 2x$

Its A.E. is $D^2+16=0 \Rightarrow D = \pm 4i$

C.F. $= C_1\cos 4x + C_2\sin 4x$

Let $y_1 = \cos 4x$; $y_2 = \sin 4x$; $X = 32\sec 2x$

Now
$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos 4x & \sin 4x \\ -4\sin 4x & 4\cos 4x \end{vmatrix} = 4(\cos^2 4x + \sin^2 4x) = 4$$

and
$$\text{P.I.} = uy_1 + vy_2 \quad \text{...(3)}$$

Where
$$u = -\int \frac{y_2 X}{W}dx = -\int \frac{\sin 4x \cdot 32\sec 2x\, dx}{4}$$

$$= -8\int \frac{2\sin 2x\cos 2x}{\cos 2x}dx = 16\frac{\cos 2x}{2}$$

$$= 8e^{3x}\frac{1}{D^2+4}\sin 4x + \frac{1}{D^2-6D+13}e^{(\log 2)x}$$

$$= 8e^{3x}\frac{1}{-4^2+4}\sin 4x + \frac{1}{(\log 2)^2-6\log 2+13}2^x$$

$$= -\frac{2}{3}e^{3x}\sin 4x + \frac{1}{(\log 2)^2-6\log 2+13}$$

$y = C.F. + P.I.$

Q 22. Solve $y''-6y'+9y=\frac{e^{3x}}{x^2}$ by variation of parameter method.

(PTU, May 2006)

Ans. The given diff. eq. be $y''-6y'+9y=\frac{e^{3x}}{x^2}$

Its A.E be $D^2-6D+9=0 \Rightarrow D=3,3$

$C.F. = (C_1+C_2x)e^{3x}$

Let $y_1 = e^{3x}$; $y_2 = xe^{3x}$, $X = \frac{e^{3x}}{x^2}$

Let its $P.I. = uy_1 + vy_2$

where $u = -\int \frac{y_2X}{w}dx$; $v = \int \frac{y_1X}{w}dx$

$$w = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{3x} & xe^{3x} \\ 3e^{3x} & e^{3x}(1+3x) \end{vmatrix} = e^{6x}(1+3x-3x) = e^{6x}$$

$$u = -\int \frac{xe^{3x}.e^{3x}}{x^2e^{6x}}dx = -\log x$$

$$v = \int \frac{e^{3x}.e^{3x}}{x^2e^{6x}}dx = -\frac{1}{x}$$

$P.I. = -e^{3x}\log x - \frac{1}{x}x.e^{3x}$

$y = (C_1+C_2x)e^{3x} - e^{3x}\log x - e^{3x} = (C_3+C_2x)e^{3x} - e^{3x}\log x$

where $C_3 = C_1 - 1$

Q 23. Solve $x^2y''-4xy'+8y=4x^3+2\sin(\log x)$ (PTU, May 2006)

Ans. The given diff. eq. can be written as

$(x^2D^2-4xD+8)y = 4x^3+2\sin(\text{Log } x)$ Where $D = \frac{d}{dx}$

It is of Cauchy's form

put $x = e^z \Rightarrow \log x = z$

i.e. $xD = \theta$, $x^2D^2 = \theta(\theta-1)$ where $\theta = \frac{d}{dz}$

$[\theta(\theta-1)-4\theta+8]y = 4e^{3z}+2\sin z$

$(\theta^2-5\theta+8)y = 4e^{3z}+2\sin z$

Its A.E. be $\theta^2-5\theta+8=0 \Rightarrow \theta = \frac{5\pm\sqrt{7}i}{2}$

$$C.F. = e^{\frac{5}{2}z}\left[C_1\cos\frac{\sqrt{7}}{2}z + C_2\sin\frac{\sqrt{7}}{2}z\right]$$

and

$$P.I. = 4.\frac{1}{\theta^2-5\theta+8}e^{3z} + 2\frac{1}{\theta^2-5\theta+8}\sin z$$

$$= 4.\frac{e^{3z}}{2} + 2\frac{1}{-5\theta+7}\sin z = 2e^{3z} + \frac{2(-5\theta-7)}{25\theta^2-49}\sin z$$

$$= 2e^{3z} + \frac{1}{37}(5\cos z + 7\sin z)$$

$$y = x^{5/2}\left[C_1\cos\left(\frac{\sqrt{7}}{2}\log x\right) + C_2\sin\left(\frac{\sqrt{7}}{2}\text{Log } x\right)\right] + 2x^3 + \frac{1}{37}[5\cos(\log x) + 7\sin(\log x)]$$

Q 24. Solve $(D^2+D+1)y = (1+\sin x)^2$ where $D = \frac{d}{dx}$ (PTU, May 2007)

Solution. The given diff. equation be

$$(D^2+D+1)y = (1+\sin x)^2 \text{ where } D = \frac{d}{dx}$$

Its A.E. is given by $D^2+D+1=0$

$$\Rightarrow D = \frac{-1\pm\sqrt{1-4}}{2} = \frac{-1\pm\sqrt{3}i}{2}$$

$$C.F. = e^{-x/2}\left[c_1\cos\left(\frac{\sqrt{3}}{2}x\right) + c_2\sin\left(\frac{\sqrt{3}}{2}x\right)\right]$$

$$P.I. = \frac{1}{D^2+D+1}(1+\sin x)^2$$

$$= \frac{1}{D^2+D+1}(1+\sin^2 x + 2\sin x)$$

Disc $= a_1^2 - 4a_2 = 0$ Hence $\lambda = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2}}{2} = \frac{-a_1}{2}, \frac{-a_1}{2}$

Now $y = (c_1 + c_2 x)e^{\lambda x}$ be the solution of $\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_2 y = 0$ ....(1)

if this sol. satisfies it

$\frac{dy}{dx} = (c_1 + c_2 x)e^{\lambda x}\lambda + e^{\lambda x} c_2$; $\frac{d^2y}{dx^2} = [\lambda c_1 + \lambda c_2 x + c_2]e^{\lambda x}\lambda + e^{\lambda x}\lambda c_2$

$$\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_2 y = [\lambda^2 c_1 + \lambda^2 c_2 x + \lambda c_2 + \lambda c_2 + a_1 c_1 \lambda + a_1 c_2 \lambda x + a_1 c_2 + a_2 c_1 + a_2 c_2 x]e^{\lambda x}$$

$$= e^{\lambda x}[c_1(\lambda^2 + a_1\lambda) + c_2 x(\lambda^2 + a_1\lambda) + 2\lambda c_2 + a_1 c_2 + a_2 c_1 + a_2 c_2 x]$$

$$= e^{\lambda x}[-a_2 c_1 - a_2 c_2 x + 2\lambda c_2 + a_1 c_2 + a_2 c_1 + a_2 c_2 x] \text{ (using (*))}$$

$$= e^{\lambda x}[2\lambda c_2 + a_1 c_2] \text{ since } \lambda = \frac{-a_1}{2}$$

$$= e^{\frac{-a_1}{2}x}[-a_1 c_2 + a_1 c_2] = 0$$

Q 36. Find the general solution of the equation $y'' + 3y' + 2y = 2e^x$, using method of variation of parameters. (PTU, May 2006)

Ans. Its A.E. be $D^2 + 3D + 2 = 0 \Rightarrow D = -1, -2$

C.F. $= (C_1 e^{-x} + C_2 e^{-2x})$

Let $y_1 = e^{-x}$; $y_2 = e^{-2x}$; $X = 2e^x$

Let its P.I. $= uy_1 + vy_2$

where $u = -\int \frac{y_2 X\,dx}{w}$; $v = \int \frac{y_1 X}{w}dx$ and $w = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

$$w = \begin{vmatrix} e^{-x} & e^{-2x} \\ -e^{-x} & -2e^{-2x} \end{vmatrix} = -2e^{-3x} + e^{-3x} = -e^{-3x}$$

$$u = -\int \frac{e^{-2x} 2e^x}{-e^{-3x}} dx = 2\int e^{2x} dx = \frac{2e^{2x}}{2} = e^{2x}$$

$$v = \int \frac{e^{-x} 2e^x\, dx}{-e^{-3x}} = -2\frac{e^{3x}}{3}$$

$$\text{P.I.} = e^{2x}e^{-x} - \frac{2}{3}e^{3x}.e^{-2x} = e^x - \frac{2}{3}e^x = \frac{1}{3}e^x$$

$$y = c_1 e^{-x} + c_2 e^{-2x} + \frac{1}{3}e^x$$

Q 37. Solve $(D^4 - 1)y = \cos x \cosh y$. (PTU, Dec. 2006)

Ans. Now $(D^4 - 1)y = \cos x \cosh y$

Its A.E. is $D^4 - 1 = 0 \Rightarrow D = \pm 1, \pm i$

C.F. $= C_1 e^x + C_2 e^{-x} + C_3 \sin x + C_4 \cos x$

$$\text{P.I.} = \frac{1}{D^4 - 1}\cos x \cosh y = \frac{1}{2}\frac{1}{D^4 - 1}\cos x(e^x + e^{-x})$$

$$= \frac{1}{2}\left[\frac{1}{D^4 - 1}e^x \cos x + \frac{1}{D^4 - 1}e^{-x}\cos x\right]$$

$$= \frac{1}{2}\left[e^x \frac{1}{(D+1)^4 - 1}\cos x + e^{-x}\frac{1}{(D-1)^4 - 1}\cos x\right]$$

$$= \frac{1}{2}\left[e^x \frac{1}{D^4 + 4D^3 + 6D^2 + 4D}\cos x + e^{-x}\frac{1}{D^4 - 4D^3 + 6D^2 - 4D}\cos x\right]$$

$$= \frac{1}{2}\left[e^x \frac{1}{1 - 4D - 6 + 4D}\cos x + e^{-x}\frac{1}{1 + 4D - 6 - 4D}\cos x\right]$$

$$= \frac{1}{2}\left[\frac{-e^x}{5}\cos x - \frac{1}{5}e^{-x}\cos x\right] = \frac{-1}{5}\cos x\left(\frac{e^x + e^{-x}}{2}\right)$$

$$= \frac{-1}{5}\cos x \cosh y$$

$y = \text{C.F.} + \text{P.I.}$

Q 38. Using method of undertermined coefficient solve

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 4y = 2x^2 + 3e^{-x}$$ (PTU, Dec. 2006)

Ans. The given diff. eq. be $(D^2 + 2D + 4)y = 2x^2 + 3e^{-x}$ ....(1)

Its A.E. be $D^2 + 2D + 4 = 0 \Rightarrow D = \frac{-2 \pm \sqrt{4 - 16}}{2} = \frac{-2 \pm 2\sqrt{3}i}{2}$

$\Rightarrow$ $D = -1 \pm \sqrt{3}i$

C.F. $= e^{-x}[C_1 \cos\sqrt{3}x + C_2 \sin\sqrt{3}x]$

Let us assume the P.I. $= y = C_1 x^2 + C_2 x + C_3 + C_4 e^{-x}$

$\Rightarrow$ $Dy = 2C_1 x + C_2 - C_4 e^{-x}$

$\Rightarrow$ $D^2 y = 2C_1 + C_4 e^{-x}$

eq (1) becomes

$2C_1 + C_4 e^{-x} + 4C_1 x + 2C_2 - 2C_4 e^{-x} + 4C_1 x^2 + 4C_2 x + 4C_3 + 4C_4 e^{-x}$

$$= e^{-x/2}\frac{1}{\left[\left(D-\frac{1}{2}\right)^4+\left(D-\frac{1}{2}\right)^2+1\right]}\cos\left(\frac{\sqrt{3}}{2}x\right)$$

$$= e^{-x/2}\frac{1}{\left[D^4-2D^3+\frac{3}{2}D^2-\frac{1}{2}D+\frac{1}{16}+D^2-D+\frac{5}{4}\right]}\left(\cos\frac{\sqrt{3}}{2}x\right)$$

$$= e^{-x/2}\frac{1}{\left[\left(\frac{-3}{4}\right)\left(\frac{-3}{4}\right)-2D\left(\frac{-3}{4}\right)+\frac{5}{2}\left(\frac{-3}{4}\right)-\frac{3}{2}D+\frac{21}{16}\right]}\cos\frac{\sqrt{3}}{2}x$$

$$= e^{-x/2}\frac{1}{\frac{9}{16}-\frac{15}{8}+\frac{21}{16}}\cos\frac{\sqrt{3}}{2}x \quad \text{(Case of failure as denominator = 0)}$$

$$= e^{-x/2}\frac{1}{4D^3-6D^2+5D-\frac{3}{2}}\cos\frac{\sqrt{3}}{2}x$$

$$= e^{-x/2}\frac{1}{4D\left(\frac{-3}{4}\right)-6\left(\frac{-3}{4}\right)+5D-\frac{3}{2}}\cos\frac{\sqrt{3}}{2}x$$

$$= e^{-x/2}\frac{1}{2D+3}\cos\frac{\sqrt{3}}{2}x$$

$$= e^{-x/2}\frac{2D-3}{4D^2-9}\cos\frac{\sqrt{3}}{2}x$$

$$= e^{-x/2}\frac{(2D-3)}{4\left(\frac{-3}{4}\right)-9}\cos\frac{\sqrt{3}}{2}x = \frac{e^{-x/2}}{-12}\left[-\sqrt{3}\sin\frac{\sqrt{3}}{2}x-3\cos\frac{\sqrt{3}}{2}x\right]$$

C.S. $= y =$ C.F + P.I

Q 33. Prove that $\frac{1}{f(D)}\sin(ax) = \frac{1}{f(-a^2)}\sin ax$; $f(-a^2) \neq 0$. **(PTU, Dec. [illegible])**

Ans. We know that

$D(\sin ax) = a\cos ax$

$D^2(\sin ax) = -a^2\sin ax = (-a^2)\sin ax$

$D^3(\sin ax) = -a^3\cos ax$

$D^4(\sin ax) = a^4\sin ax = (-a^2)^2\sin ax$

i.e. $(D^2)^2(\sin ax) = (-a^2)^2\sin ax$

$(D^2)^n(\sin ax) = (-a^2)^n\sin ax$

or In general $f(D^2)\sin ax = f(-a^2)\sin ax$

operate both sides by $\frac{1}{f(D^2)}$

$$\frac{1}{f(D^2)}(f(D^2)\sin ax) = \frac{1}{f(D^2)}(f(-a^2)\sin ax)$$

$$\Rightarrow \quad \sin ax = f(-a^2)\left[\frac{1}{f(D^2)}\sin ax\right]$$

$$\Rightarrow \quad \frac{1}{f(-a^2)}\sin ax = \frac{1}{f(D^2)}\sin ax$$

Q 34. Solve $\frac{d^2y}{dx^2}+\frac{1}{x}\frac{dy}{dx} = \frac{12\log x}{x^2}$ **(PTU, Dec. [illegible])**

Ans. Given $\frac{d^2y}{dx^2}+\frac{1}{x}\frac{dy}{dx} = \frac{12\log x}{x^2} \Rightarrow x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx} = 12\log x$

$\Rightarrow \quad (x^2D^2+xD)y = 12\log x$; where $D = \frac{d}{dx}$ which is of Cauchy's form

put $x = e^t \Rightarrow \log x = t$

$xD = \theta$; $x^2D^2 = \theta(\theta-1)$ where $\theta = \frac{d}{dt}$

$[\theta(\theta-1)+\theta]y = 12t \Rightarrow \theta^2y = 12t$

$\Rightarrow \quad y = 2t^3 + C_1t + C_2$

$\Rightarrow \quad y = 2(\log x)^3 + C_1\log x + C_2$

where C_1, C_2 are arbitary constants

Q 35. If two roots of the auxiliary equation $\lambda^2 + a_1\lambda + a_2 = 0$ are real and equal, then prove that $y = (c_1 + c_2x)e^{\lambda x}$ is a solution of the equation $y'' + a_1y' + a_2y = 0$ **(PTU, May [illegible])**

Ans. If the roots of A.E. $\lambda^2 + a_1\lambda + a_2 = 0$ ......[illegible] are real and distinct

$$\theta = -1, \frac{2 \pm \sqrt{8}}{2} \text{ i.e. } \theta = -1, 1 \pm i$$

$$C.F. = C_1 e^{-t} + (C_2 \cos t + C_3 \sin t)e^{t}$$

$$P.I. = 10 \frac{1}{\theta^3 - \theta^2 + 2}\left[e^{t} + e^{-t}\right] = 10\left[\frac{1}{\theta^3 - \theta^2 + 2} e^{t} + \frac{1}{\theta^3 - \theta^2 + 2} e^{-t}\right]$$

$$= 10\left[\frac{e^{t}}{1 - 1 + 2} + t \cdot \frac{1}{3\theta^2 - 2\theta} e^{-t}\right]$$

$$= 10\left[\frac{e^{t}}{2} + \frac{te^{-t}}{5}\right]$$

$$y = \frac{C_1}{x} + x\,(C_2 \cos(\log x) + C_3 \sin(\log x)) + 5x + 2\,\frac{\log x}{x}$$

Q 30. Using method of variation of parameters,

solve, $\frac{d^2y}{dx^2} + y = \sec x$ **(PTU, Dec. 200[illegible])**

Ans. The given diff. eq. can be written as $(D^2 + 1)\,y = \sec x$

Its A.E. is $D^2 + 1 = 0 \Rightarrow D = \pm i$

$$C.F. = C_1 \cos x + C_2 \sin x$$

Let $y_1 = \cos x,\ y_2 = \sin x,\ X = \sec x$

so that its particular Integral be $uy_1 + vy_2$

where

$$u = -\int \frac{y_2 X}{w}\,dx\ ;\ v = +\int \frac{y_1 X}{w}\,dx$$

$$w = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$$

$$u = -\int \sin x \sec x\,dx = \log|\cos x|$$

$$v = \int \cos x \sec x\,dx = x$$

$$P.I. = \sin x \log|\cos x| + x \sin x$$

$$y = C.F. + P.I.$$

Q 31. Solve

$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = \log x \sin(\log x)$ **(PTU, Dec. 2009, 200[illegible])**

Ans. The given diff. eq. can be written as

$(x^2D^2 + xD + 1)\,y = \log x \sin(\log x)$ ...(1), $D = \frac{d}{dx}$

which is of Cauchy's linear diff. eq.

put $x = e^t \Rightarrow \log x = t$

s.t. $xD = \theta,\ x^2D^2 = \theta(\theta - 1)$ where $\theta = \frac{d}{dt}$, eq (1) gives

$\Rightarrow$ $[\theta(\theta - 1) + \theta + 1]\,y = t \sin t$

$$(\theta^2 + 1)\,y = t \sin t$$

Its A.E. is $\theta^2 + 1 = 0 \Rightarrow \theta = \pm i$

$\Rightarrow$ $C.F. = C_1 \cos t + C_2 \sin t$

$$I = P.I. = \frac{1}{\theta^2 + 1}\,t \sin t = t \cdot \frac{1}{\theta^2 + 1}(\sin t) - \frac{1}{\theta^2 + 1}\left(2\theta\left(\frac{1}{\theta^2 + 1}(\sin t)\right)\right)$$

$$= t \cdot t\,\frac{1}{2\theta} \sin t - \frac{1}{\theta^2 + 1}\left(2\theta\left(t \cdot \frac{1}{2\theta}(\sin t)\right)\right)$$

$$= \frac{-t^2}{2}\cos t - \frac{2\theta}{\theta^2 + 1}\left(\frac{-t\cos t}{2}\right)$$

$$= \frac{-t^2}{2}\cos t + \frac{1}{\theta^2 + 1}(-t \sin t + \cos t)$$

$\Rightarrow$ $$2I = \frac{-t^2}{2}\cos t + \frac{t}{2}\sin t \Rightarrow I = \frac{-t^2}{4}\cos t + \frac{t}{4}\sin t$$

$$y = C_1 \cos t + C_2 \sin t - \frac{t^2}{4}\cos t + \frac{t}{4}\sin t$$

$\Rightarrow$ $$y = C_1 \cos(\log x) + C_2 \sin(\log x) - \frac{(\log x)^2}{4}\cos(\log x) + \frac{\log x}{4}\sin(\log x)$$

Q 32. Solve the differential equation :

$(D^4 + D^2 + 1)\,y = e^{-x/2} \cdot \cos\left(x\sqrt{3}/2\right)$ **(PTU, May 2004)**

Ans. Its A.E. is given by $D^4 + D^2 + 1 = 0 \Rightarrow (D^2 - D + 1)(D^2 + D + 1) = 0$

$$D = \frac{1 \pm \sqrt{3}i}{2}, \frac{-1 \pm \sqrt{3}i}{2}$$

$$C.F. = \left(C_1 \cos\frac{\sqrt{3}}{2}x + C_2 \sin\frac{\sqrt{3}}{2}x\right)e^{\frac{x}{2}} + e^{-x/2}\left(C_3 \cos\frac{\sqrt{3}}{2}x + C_4 \sin\frac{\sqrt{3}}{2}x\right)$$

$$P.I. = \frac{1}{D^4 + D^2 + 1}\,e^{-x/2}\cos\left(\frac{\sqrt{3}}{2}x\right)$$

Q 43. What do you understand by complementary function? Explain. (PTU, Dec. 2009)

Solution. The function obtained from the roots of the auxiliary equation is known as complementary function (C.F.). It depends upon the nature of roots and contains as many as arbitrary constants as the order of the differential equation.

Q 44. Solve the Cauchy-Euler equation :
$x^2y'' - xy' + 2y = x\log_e x, x > 0.$ (PTU, May 2010)

Solution. It is of Cauchy's equation from and in symbolic form, it can be written as,

$(x^2D^2 - xD + 2)y = x\log x$ ...(1)

put, $x = e^z \Rightarrow \log x = z$, $xD = \theta$, $x^2D^2 = \theta(\theta - 1)$, $\theta = \frac{d}{dz}$

Therefore, eqn. (1) gives ; $[\theta(\theta - 1) - \theta + 2]y = z.e^z$

$(\theta^2 - 2\theta + 2)y = ze^z$

Its A.E. is, $\theta^2 - 2\theta + 2 = 0 \Rightarrow \theta = \frac{2 \pm \sqrt{4-8}}{2} = 1 \pm i$

C.F. $= [c_1 \cos z + c_2 \sin z]e^z$

and P.I. $= \frac{1}{\theta^2 - 2\theta + 2}e^z.z = e^z\frac{1}{(\theta+1)^2 - 2(\theta+1) + 2}.z$

$= e^z.\frac{1}{\theta^2 + 1}.z = e^z.[1 + \theta^2]^{-1}(z)$

$= e^z[1 - \theta^2 \ldots](z) = ze^z$

Therefore, complete solution $y = [c_1 \cos(\log x) + c_2 \sin(\log x)]x + x\log x\ x.$

Q 45. Solved the equation $\frac{d^4y}{dx^4} + 2\frac{d^2y}{dx^2} + y = 0.$ (PTU, Dec. 2010)

Solution. The given differential equation can be written as

$(D^4 + 2D^2 + 1)y = 0 ; D = \frac{d}{dx}$

Its A.E. is given by,

$D^4 + 2D^2 + 1 = 0 \Rightarrow (D^2 + 1)^2 = 0$

$\Rightarrow D = \pm i, \pm i$

and complete solution is given by

$y = [(C_1 + C_2x)\cos x + (C_3 + C_4x)\sin x].$

Q 46. Use method of variation of parameters to solve $\frac{d^2y}{dx^2} + 4y = \tan 2x.$ (PTU, Dec. 2010)

Solution. Its symbolic form is

$(D^2 + 4)y = \tan 2x$

Its A.E. is, $D^2 + 4 = 0 \Rightarrow D = \pm 2i$

C.F. $= c_1 \cos 2x + c_2 \sin 2x$ ...(1)

Let P.I. $= uy_1 + vy_2$

where $y_1 = \cos 2x, y_2 = \sin 2x, X = \tan 2x$

$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix} = 2$

Now $u = -\int \frac{y_2X}{W}dx = -\int \frac{\sin 2x \tan 2x}{2}dx = -\frac{1}{2}\int \frac{1 - \cos^2 2x}{\cos 2x}dx$

and $v = \int \frac{y_1X}{W}dx = \int \frac{\cos 2x \tan 2x}{2}dx = \int \frac{\sin 2x}{2}dx$

$= -\frac{\cos 2x}{4}$

P.I. $= \cos 2x\left[-\frac{1}{4}\log(\sec 2x + \tan 2x) + \frac{1}{4}\sin 2x\right] - \frac{\cos 2x}{4}\sin 2x$

$= -\frac{\cos 2x}{4}(\sec 2x + \tan 2x)$

C.S. $= y = c_1 \cos 2x + c_2 \sin 2x - \frac{1}{4}\cos 2x \log(\sec 2x + \tan 2x)$

Q 47. Solve : $(D^2 + 1)y = \text{cosec}\ x.\cot x.$ (PTU, May 2011)

Solution. We use variation of parameter method

C.F $= C_1 \cos x + C_2 \sin x$

Let P.I $= uy_1 + vy_2$ ...(1)

Where, $y_1 = \cos x ; y_2 = \sin x ; X = \text{cosec}\ x \cot x$

$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1 \neq 0$

$u = -\int \frac{y_2X}{W}dx = -\int \frac{\sin x.\text{cosec}\ x \cot x}{1}dx = -\log|\sin x|$

$v = \int \frac{y_1X}{W}dx = \int \cos x\ \text{cosec}\ x \cot x\,dx = \int (\text{cosec}^2 x - 1)dx$

$= -\cot x - x$

$\therefore$ eqn (1) gives ;

P.I. $= -\log|\sin x|.\cos x + (-\cot x - x)\sin x$

C.S $= y = C_1 \cos x + C_2 \sin x - \cos x \log|\sin x| - x \sin x - \cos x$

i.e. $y = C_1' \cos x + C_2 \sin x - \cos x \log|\sin x| - x \sin x$

Where $C_1' = (C_1 - 1)$

Q 48. Solve $\frac{d^4x}{dt^4} + 4x = 0.$ (PTU, May 2011)

Solution. The given diff. eqn. can be written as

$(D^4 + 4)x = 0 ; D = \frac{d}{dt}$

Hence the C.S is $y = (C_1 + C_2 x) e^{2x} + 4x^2 e^{2x} + \cos 2x + 2x^2 + 4x + 3$

(b) Given equation is Cauchy's homogeneous linear equation

$x = e^z$ i.e. $z = \log x$

$\therefore \quad xD = \theta \; ; \; x^2 D^2 = \theta(\theta - 1) \; ; \; x^3 D^3 = \theta(\theta - 1)(\theta - 2)$

Where $\theta = \frac{d}{dz}$

Substituting these values in the given equation, it reduces to

$[\theta(\theta - 1)(\theta - 2) + 2\theta(\theta - 1) + 2] y = 10(e^z + e^{-z})$

$(\theta^3 - \theta^2 + 2) y = 10(e^z + e^{-z})$

Which is a linear equation with constants coefficients.

Its A.E. is $\theta^3 - \theta^2 + 2 = 0$ or $(\theta + 1)(\theta^2 - 2\theta + 2) = 0$

$$\theta = -1, \frac{2 \pm \sqrt{4 - 8}}{2} = -1, 1 \pm i$$

$$C.F. = C_1 e^{-z} + e^z (C_2 \cos z + C_3 \sin z)$$

$$= \frac{C_1}{x} + x[C_2 \cos(\log x) + C_3 \sin(\log x)]$$

and

$$P.I. = 10 \frac{1}{\theta^3 - \theta^2 + 2}(e^z + e^{-z}) = 10\left(\frac{1}{\theta^3 - \theta^2 + 2} e^z + \frac{1}{\theta^3 - \theta^2 + 2} e^{-z}\right)$$

$$\therefore P.I. = 10\left(\frac{1}{1^3 - 1^2 + 2} e^z + z \frac{1}{3\theta^2 - 2\theta} e^{-z}\right) = 10\left(\frac{1}{2} e^z + z \frac{1}{3(-1)^2 - 2(-1)} e^{-z}\right)$$

$$= 5e^z + 2ze^{-z} = 5x + \frac{2}{x} \log x$$

Hence the C.S is $y = \frac{C_1}{x} + x[C_2 \cos(\log x) + C_3 \sin(\log x)] + 5x + \frac{2}{x} \log x$

Q 42. Solve :
$(D^2 - 1) y = e^{2x} \cos 2x - e^{2x} \sin 3x$ **(PTU, May 2009)**
using method of undetermined coefficients.

Ans. $(D^2 - 1) y = e^{2x} \cos 2x - e^{2x} \sin 3x$ ....(1)

$C.F. = C_1 e^x + C_2 e^{-x}$

Trial solution is

$$y = e^{2x}(A_0 \sin 2x + A_1 \cos 2x) + e^{2x}(A_2 \sin 3x + A_3 \cos 3x) \quad ...(2)$$

$$\frac{dy}{dx} = e^{2x}[2A_0 \cos 2x - 2A_1 \sin 2x] + 3e^{3x}[A_0 \sin 2x + A_1 \cos 2x]$$
$$+ e^{2x}[3A_2 \cos 3x - 3A_3 \sin 3x] + [A_2 \sin 3x + A_3 \cos 3x] 2e^{2x}$$

$$\frac{d^2y}{dx^2} = e^{2x}[-4A_0 \sin 2x - 4A_1 \cos 2x] + [2A_0 \cos 2x - 2A_1 \sin 2x](3e^{3x})$$
$$+ 9e^{3x}[A_0 \sin 2x + A_1 \cos 2x] + 3e^{3x}[2A_0 \cos 2x - 2A_1 \sin 2x]$$
$$+ 2e^{2x}[3A_2 \cos 3x - 3A_3 \sin 3x] + e^{2x}[-9A_2 \sin 3x - 9A_3 \cos 3x]$$
$$+ 4e^{2x}[A_2 \sin 3x + A_3 \cos 3x] + 2e^{2x}[3A_2 \cos 3x - 3A_3 \sin 3x]$$

$$\frac{d^2y}{dx^2} = e^{2x}[-4A_0 \sin 2x - 4A_1 \cos 2x + 6A_0 \cos 2x - 6A_1 \sin 2x$$
$$+ 9A_0 \sin 2x + 9A_1 \cos 2x + 6A_0 \cos 2x - 6A_1 \sin 2x]$$
$$+ e^{2x}[6A_2 \cos 3x - 6A_3 \sin 3x - 9A_2 \sin 3x - 9A_3 \cos 3x + 4A_2 \sin 3x + 4A_3 \cos 3x + 6A_2 \cos 3x - 6A_3 \sin 3x]$$

$$\frac{d^2y}{dx^2} = e^{2x}[5A_0 \sin 2x + 5A_1 \cos 2x + 12A_0 \cos 2x - 12A_1 \sin 2x]$$
$$+ e^{2x}[12A_2 \cos 3x - 12A_3 \sin 3x - 5A_2 \sin 3x - 5A_3 \cos 3x] \quad ...(3)$$

Put in (1) the value of (2) and (3), we get

$e^{2x}[4A_0 \sin 2x + 4A_1 \cos 2x + 12A_0 \cos 2x - 12A_1 \sin 2x] + e^{2x}[12A_2 \cos 3x - 12A_3 \sin 3x - 6A_2 \sin 3x - 6A_3 \cos 3x]$

$$= e^{2x} \cos 2x - e^{2x} \sin 3x$$

equating the coefficients of like terms, we have

$e^{2x} \cos 2x : 4A_1 + 12A_0 = 1$ ...(i)

$e^{2x} \sin 3x : -12A_3 - 6A_2 = -1$ ...(ii)

$e^{2x} \sin 2x : 4A_0 - 12A_1 = 0 \Rightarrow A_0 = 3A_1$ ...(iii)

$e^{2x} \cos 3x : 12A_2 - 6A_3 = 0 \; ; \; A_3 = 2A_2$ ...(iv)

from (i) : $4A_1 + 12A_0 = 1$

$\Rightarrow \quad 4A_1 + 36A_1 = 1$ (by (iii))

$$A_1 = \frac{1}{40}$$

by (iii), we get

$$A_0 = 3A_1 = \frac{3}{40}$$

from (ii), we get

$12A_3 + 6A_2 = 1$

$24A_2 + 6A_2 = 1$ (by iv)

$$A_2 = \frac{1}{30}$$

(by iv) $\quad A_3 = 2A_2 = \frac{2}{30}$

Now substituting all these value of A_0, A_1, A_2 and A_3 in (2), we get

$$P.I. = e^{2x}\left[\frac{3}{40} \sin 2x + \frac{1}{40} \cos 2x\right] + e^{2x}\left[\frac{1}{30} \sin 3x + \frac{2}{30} \cos 3x\right]$$

$$= \frac{e^{2x}}{40}(3 \sin 2x + \cos 2x) + \frac{e^{2x}}{30}(\sin 3x + 2 \cos 3x)$$

$$P.I. = \frac{e^{2x}}{30}(2 \cos 3x + \sin 3x) + \frac{e^{2x}}{40}(3 \sin 2x + \cos 2x)$$

Now complete solution is given by

$y = P.I. + C.F.$

$$y = C_1 e^x + C_2 e^{-x} + \frac{e^{2x}}{30}(2 \cos 3x + \sin 3x) + \frac{e^{2x}}{40}(\cos 2x + 3 \sin 2x)$$

$= 2x^2 + 3e^{-x}$

Comparing the corresponding coeff. on both sides

$$\text{coeff. of } x^2;\ 4C_1 = 2 \Rightarrow C_1 = \frac{1}{2}$$

$$\text{coeff. of } x;\ 4C_1 + 4C_2 = 0 \Rightarrow C_2 = \frac{-1}{2}$$

Constant term; $2C_1 + 2C_2 + 4C_3 = 0 \Rightarrow C_3 = 0$

Coeff of e^{-x}; $C_4 - 2C_4 + 4C_4 = 3 \Rightarrow C_4 = 1$

$$\text{P.I.} = \frac{1}{2}x^2 - \frac{1}{2}x + e^{-x}$$

$$\text{C.S.} = y = \text{C.F.} + \text{P.I.}$$

Q 39. The complementary part of the differential equation $x^2 y'' - xy' + y = \log x$ is ...... (PTU, May 2009)

Solution. $x^2 \frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = \log x$

Given differential equation is a Cauchy's homogeneous linear differential equation

Put $x = e^z \Rightarrow z = \log x$

$$x\frac{dy}{dx} = \theta y,\ x^2\frac{d^2y}{dx^2} = \theta(\theta - 1)y;\ \text{where } \theta = \frac{d}{dz}$$

$[\theta(\theta - 1)y - \theta y + y] = z$

$(\theta^2 - 2\theta + 1)y = z$

and A.E is $(\theta^2 - 2\theta + 1) = 0$

$\theta = 1, 1$

$\therefore$ C.F. $= (C_1 + C_2 z)e^z = (C_1 + C_2 \log x)x$

Q 40. The particular integral of $(D^2 + a^2)y = \sin ax$ is

(i) $\frac{-x}{2a}\cos ax$ (ii) $\frac{x}{2a}\cos ax$ (iii) $\frac{-ax}{2a}\cos ax$ (iv) $\frac{ax}{2a}\cos ax$

(PTU, May 2009)

Solution.

$$\text{P.I.} = \frac{1}{D^2 + a^2}\sin ax$$

$$= \frac{1}{-a^2 + a^2}\sin ax\ (D^2 = -a^2)$$

$$= \frac{\sin ax}{0}\ \text{(Case of failure)}$$

$$\text{P.I.} = \frac{x}{\frac{d}{dD}(D^2 + a^2)}\sin ax$$

$$= \frac{x}{2D}\sin ax$$

$$= \frac{x}{2}\int \sin ax\, dx$$

$$\text{P.I.} = -\frac{x}{2a}\cos ax$$

Q 41. Solve the following :

(a) $(D - 2)^2 y = 8(e^{2x} + \sin 2x + x^2)$.

(b) $x^3 y''' + 2x^2 y'' + 2y = 10\left(x + \frac{1}{x}\right)$. (PTU, May 2009)

Solution. (a) A.E is $(D - 2)^2 = 0$ i.e. $D = 2, 2$

C.F. $= (C_1 + C_2 x)e^{2x}$

and

$$\text{P.I.} = \frac{1}{(D - 2)^2}[8(e^{2x} + 8\sin 2x + x^2)]$$

$$= 8\left[\frac{1}{(D-2)^2}e^{2x} + \frac{1}{(D-2)^2}\sin 2x + \frac{1}{(D-2)^2}x^2\right]$$

Now, $\frac{1}{(D-2)^2}e^{2x}$ Put $D = 2$, case of failure

$$= x\frac{1}{2(D-2)}e^{2x}\ \text{[Put } D = 2\text{; case of failure]}$$

$$= x^2\frac{1}{2}e^{2x} = \frac{x^2}{2}e^{2x}$$

$$\frac{1}{(D-2)^2}\sin 2x = \frac{1}{D^2 - 4D + 4}\sin 2x = \frac{1}{-4 - 4D + 4}\sin 2x \quad [\text{Put } D^2 = -2^2]$$

$$= \frac{-1}{4D}\sin 2x = \frac{-1}{4}\int \sin 2x\, dx = \frac{-1}{4}\left(-\frac{\cos 2x}{2}\right) = \frac{1}{8}\cos 2x$$

Now

$$\frac{1}{(D-2)^2}x^2 = \frac{1}{(2-D)^2}x^2 = \frac{1}{4\left(1 - \frac{D}{2}\right)^2}x^2 = \frac{1}{4}\left(1 - \frac{D}{2}\right)^{-2}x^2$$

$$= \frac{1}{4}\left[1 - 2\left(-\frac{D}{2}\right) + \frac{(-2)(-3)}{2}\left(\frac{D}{2}\right)^2 ......\right]x^2$$

$$= \frac{1}{4}\left[1 + D + \frac{3}{4}D^2 + ...\right]x^2 = \frac{1}{4}\left(x^2 + 2x + \frac{3}{2}\right)$$

$$\therefore \text{P.I.} = 8\left[\frac{x^2}{2}e^{2x} + \frac{1}{8}\cos 2x + \frac{1}{4}\left(x^2 + 2x + \frac{3}{2}\right)\right] = 4x^2e^{2x} + \cos 2x + 2x^2 + 4x + 3$$

Here, $u = -\int \frac{y_2 X}{W} dx$ and $v = \int \frac{y_1 X}{W} dx$

Here, $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-x} & e^{-2x} \\ -e^{-2x} & -2e^{-2x} \end{vmatrix} = -2e^{-3x} + e^{-3x} = -e^{-3x}$

$$u = -\int \frac{e^{-2x} \sin e^x}{-e^{-3x}} dx = \int e^x \sin e^x dx$$

i.e. $u = -\cos e^x$

and $v = \int \frac{e^{-x} \sin e^x}{-e^{-3x}} dx = -\int e^{2x} \sin e^x dx$ (put $e^x = t$, $e^x dx = dt$)

$$= -\int t \sin t \, dt = -[-t \cos t + \sin t] = [e^x \cos e^x - \sin e^x]$$

$$P.I = -e^{-x} \cos e^x + e^{-2x} [e^x \cos e^x - \sin e^x] = -e^{-2x} \sin e^x$$

$$y = C.F + P.I = C_1 e^{-x} + C_2 e^{-2x} - e^{-2x} \sin e^x$$

Q 57. Solve : $\frac{d^2y}{dx^2} + a^2 y = \sin ax$. **(PTU, Dec. 2002)**

Ans. (a) The given diff. eq. be $(D^2 + a^2) y = \sin ax$

Its A.E. be $D^2 + a^2 = 0 \Rightarrow D = \pm ai$

$$C.F. = C_1 \cos ax + C_2 \sin ax$$

$$P.I. = \frac{1}{D^2 + a^2} \sin ax = x \cdot \frac{1}{2D} \sin ax \text{ [by using case of failure]}$$

$$= -\frac{x}{2a} \cos ax$$

$$y = C.F. + P.I. = C_1 \cos ax + C_2 \sin ax - \frac{x}{2a} \cos ax$$

Q 58. Solve : $x^2 \frac{d^2y}{dx^2} + 3x \frac{dy}{dx} + y = \frac{1}{(1-x)^2}$ **(PTU, Dec. 2002)**

Ans. The given diff. eq. is of Cauchy's form

put $x = e^z \Rightarrow \log x = z$

s.t. $xD = \theta$, $x^2 D^2 = \theta(\theta - 1)$ where $\theta = \frac{d}{dz}$

and the given eq becomes

$$[\theta(\theta-1) + 3\theta + 1] y = \frac{1}{(1-e^z)^2} \Rightarrow [\theta^2 + 2\theta + 1] y = \frac{1}{(1-e^z)^2}$$

Its A.E. be $\theta^2 + 2\theta + 1 = 0 \Rightarrow \theta = -1, -1$

$$C.F. = (C_1 + C_2 z) e^{-z}$$

$$P.I. = \frac{1}{(\theta+1)^2} \frac{1}{(1-e^z)^2} = \frac{1}{(1+\theta)} \left[e^{-z} \int e^z \cdot \frac{1}{(1-e^z)^2} dz \right]$$

$$= \frac{1}{(1+\theta)} \left[e^{-z} \left(\frac{1}{1-e^z} \right) \right] = e^{-z} \int e^z \cdot e^{-z} \frac{1}{1-e^z} dz = e^{-z} \int \frac{1}{1-e^z} dz$$

$$= e^{-z} \int \frac{dt}{t(1-t)} \text{ put } e^z = t \Rightarrow e^z dz = dt$$

$$= e^{-z} \left[\int \left(\frac{1}{t} + \frac{1}{1-t} \right) dt \right] = e^{-z} \left[\log \frac{e^z}{1-e^z} \right]$$

$$y = (C_1 + C_2 z) e^{-z} + e^{-z} \log \frac{e^z}{1-e^z}$$

$$= (C_1 + C_2 \log x) \frac{1}{x} + \frac{1}{x} \log \frac{x}{1-x}$$

Q 59. Solve $(D^2 + 4)y = x \sin 2x$. **(PTU, Dec. 2002)**

Ans. Its A.E. be $D^2 + 4 = 0 \Rightarrow D = \pm 2i$

$$C.F. = C_1 \cos 2x + C_2 \sin 2x$$

$$P.I. = \frac{1}{D^2+4} x \sin 2x = x \cdot \frac{1}{D^2+4} \sin 2x - \frac{1}{D^2+4} \left[(2D) \left(\frac{1}{D^2+4} \sin 2x \right) \right] \quad \text{...(1)}$$

Now $\frac{1}{D^2+4} \sin 2x = x \cdot \frac{1}{2D} \sin 2x = \frac{-x \cos 2x}{4}$

∴ eq (1) gives

$$I = P.I. = \frac{-x^2}{4} \cos 2x + \frac{1}{2(D^2+4)} [\cos 2x - 2x \sin 2x]$$

$$= \frac{-x^2}{4} \cos 2x + \frac{1}{2} \frac{x}{2D} \cos 2x - 1$$

$$P.I. = I = \frac{-x^2}{8} \cos 2x + \frac{x}{16} \sin 2x$$

$$y = C.F. + P.I.$$

$$= C_1 \cos 2x + C_2 \sin 2x - \frac{x^2}{8} \cos 2x + \frac{x}{16} \sin 2x$$

Q 60. Solve the differential equation $\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 0$. **(PTU, Dec. 2020)**

Ans. Given differential eqn. in symbolic form be given by

$$(D^2 + D + 1)y = 0; D = \frac{d}{dx}$$

Its Auxiliary eqn be $m^2 + m + 1 = 0$

$$\Rightarrow \quad m = \frac{-1 \pm \sqrt{3}i}{2}$$

$$D^2(e^{ax}) = a^2 e^{ax}$$

$$D^n(e^{ax}) = a^n e^{ax}$$

$\Rightarrow$ $(P_0 D^n + P_1 D^{n-1} \ldots + P_n) e^{ax} = (P_0 a^n + P_1 a^{n-1} + \ldots P_n) e^{ax}$

$$f(D) e^{ax} = f(a) e^{ax}$$

operating both sides by $\frac{1}{f(D)}$, we get

$$\frac{1}{f(D)} [f(D) e^{ax}] = \frac{1}{f(D)} [f(a) e^{ax}]$$

$\Rightarrow$ $$e^{ax} = f(a) \left[\frac{1}{f(D)} e^{ax}\right]$$

$\Rightarrow$ $$\frac{1}{f(a)} e^{ax} = \frac{1}{f(D)} e^{ax}$$

Q 54. Solve by method of variation of parameters $\frac{d^2y}{dx^2} + y = \tan x$ **(PTU, Dec. 2004)**

Ans. The given diff. eq. can be written as $(D^2 + 1) y = \tan x$

Its A.E. is given by $D^2 + 1 = 0 \Rightarrow D = \pm i$

$$C.F. = C_1 \cos x + C_2 \sin x$$

Let the P.I. be $= uy_1 + vy_2$

where $u = -\int \frac{y_2 X}{W} dx$, $v = \int \frac{y_1 X}{W}$; $y_1 = \cos x$, $y_2 = \sin x$

where $X = \tan x$, $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$

$$u = -\int \sin x \tan x \, dx = -\int \frac{\sin^2 x}{\cos x} dx = -\int \frac{1-\cos^2 x}{\cos x} dx$$

$$= -\int \sec x \, dx + \int \cos x \, dx = -\log |\sec x + \tan x| + \sin x$$

$$v = \int \cos x \tan x \, dx = \int \sin x \, dx = -\cos x$$

$\therefore$ $y = C.F. + P.I. = C_1 \cos x + C_2 \sin x - \cos x \log |\sec x + \tan x| + \cos x \sin x - \sin x \cos x$

$$y = C_1 \cos x + C_2 \sin x - \cos x \log |\sec x + \tan x|$$

Q 55. Solve the following simultaneous differential equation

$\frac{dx}{dt} - 2y + 5x = t$, $\frac{dy}{dt} + 2x + y = 0$. Given that $x(0) = 0$, $y(0) = 0$. **(PTU, Dec. 2012)**

Solution. In symbolic form ; $(D + 5) x - 2y = t$ ...(1)

$2x + (D + 1) y = 0$ ...(2)

Multiply eqn. (1) by $(D + 1)$ and equation (2) by 2 and adding, we have

$[(D + 5)(D + 1) + 4] x = (D + 1) t$

$\Rightarrow$ $[D^2 + 6D + 9] x = 1 + t$

Its A.E. becomes, $D^2 + 6D + 9 = 0$

$\Rightarrow$ $D = -3, -3$

$$C.F. = (c_1 + c_2 t) e^{-3t}$$

and $$P.I. = \frac{1}{(D+3)^2}(1+t) = \frac{1}{9}\left[1+\frac{D}{3}\right]^{-2}(1+t)$$

$$= \frac{1}{9}\left[1 - \frac{2}{3}D + \ldots\right](1+t) = \frac{1}{9}\left[1+t-\frac{2}{3}\right] = \frac{1}{9}\left[t+\frac{1}{3}\right]$$

$\therefore$ $$x = (c_1 + c_2 t) e^{-3t} + \frac{1}{9}\left(t + \frac{1}{3}\right)$$

Thus, $$\frac{dx}{dt} = -3(c_1 + c_2 t) e^{-3t} + e^{-3t} c_2 + \frac{1}{9}$$ ...(3)

putting eqn. (3) and eqn. (4) in eqn. (1), we get

$$2y = -t - 3(c_1 + c_2 t) e^{-3t} + e^{-3t} c_2 + \frac{1}{9} + 5e^{-3t}(c_1 + tc_2) + \frac{5}{9}\left(t + \frac{1}{3}\right)$$ ...(4)

i.e. $$2y = e^{-3t}[c_2 + 2c_1 + 2c_2 t] + \left(\frac{-4}{9}t + \frac{8}{27}\right)$$ ...(5)

given when $x = 0$, $y = 0$, when $t = 0$

Therefore, from eqn. (3) ; we have

$$0 = c_1 + \frac{1}{27} \Rightarrow c_1 = -\frac{1}{27}$$

and from eqn. (4) ; we have

$$0 = c_2 + 2c_1 + \frac{8}{27} \Rightarrow c_2 = \frac{-2}{9}$$

$$x = \left[\frac{-1}{27} - \frac{2}{9}t\right] e^{-3t} + \frac{1}{27}(1 + 3t)$$

and $$y = \frac{e^{-3t}}{2}\left[\frac{-8}{27} - \frac{4}{9}t\right] + \frac{1}{2}\left(\frac{-4}{9}t + \frac{8}{27}\right)$$ is the required solution.

Q 56. Obtain the general solution of the equation $y'' + 3y' + 2y = \sin(e^x)$, by using method of variation of parameters. **(PTU, May 2012)**

Solution. The given diff. eqn can be written as

$$(D^2 + 3D + 2) y = \sin e^x$$

Its A.E is given by

$$D^2 + 3D + 2 = 0 \Rightarrow D = -1, -2$$

$$C.F = C_1 e^{-x} + C_2 e^{-2x}$$

Let its P.I be,

$$P.I = uy_1 + vy_2$$

where $y_1 = e^{-x}$ and $y_2 = e^{-2x}$ and $X = \sin e^x$

Its A.E. is $D^4 + 4 = 0 \Rightarrow D^4 + 4D^2 + 4 - 4D^2 = 0$

i.e. $(D^2 + 2)^2 - (2D)^2 = 0$

$(D^2 - 2D + 2)(D^2 + 2D + 2) = 0$

i.e. $D = \frac{2 \pm \sqrt{4-8}}{2}, \frac{-2 \pm \sqrt{4-8}}{2}$

i.e. $D = 1 \pm i, -1 \pm i$

$x = e^t\,[C_1 \cos t + C_2 \sin t] + e^{-t}\,[C_3 \cos t + C_4 \sin t].$

Q 49. Show that the two functions sin 2x, cos 2x are independent solution of $y'' + 4y = 0$. (PTU, May 2011)

Solution. The given diff. eqn can be written as $(D^2 + 4)\,y = 0$

$y = C_1 \cos 2x + C_2 \sin 2x$

Let $C_1 \cos 2x + C_2 \sin 2x = 0$ which is only possible if $C_1 = C_2 = 0$

Thus cos 2x and sin 2x are L.I.

Also $W = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix} = 2 \neq 0$

$\therefore$ W (cos 2x, sin 2x) ≠ 0. Thus the given functions are L.I.

Q 50. Solve $(D^2 - 2D + 1)\,y = xe^x \sin x$. (PTU, May 2011)

Solution. The given differential equation can be written as

$(D^2 - 2D + 1)\,y = xe^x \sin x$

Its auxiliary equation be $D^2 - 2D + 1 \Rightarrow D = 1, 1$

$C.F. = (c_1 + c_2 x)\,e^x$

and
$$P.I. = \frac{1}{D^2 - 2D + 1}\,x e^x \sin x = \frac{1}{(D-1)^2}\,e^x (x \sin x)$$

$$= e^x \frac{1}{[D+1-1]^2}\,x \sin x = e^x \cdot \frac{1}{D^2}\,x \sin x$$

$$= e^x \left[x \cdot \frac{1}{D^2} \sin x - \frac{1}{D^2}\left\{2D\left(\frac{1}{D^2} \sin x\right)\right\}\right] \qquad \text{[Rule-V]}$$

$$= e^x \left[-x \sin x + \frac{1}{D^2}(2D \sin x)\right]$$

$$= e^x\,[-x \sin x - 2 \cos x]$$

Therefore complete solution $y = C.F. + P.I.$

i.e. $y = (c_1 + c_2 x)\,e^x + e^x\,(-x \sin x - 2 \cos x).$

Q 51. Solve by method of variation of parameter the differential equation

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = e^x \tan x.$$ **(PTU, Dec. 2011)**

Solution. Its symbolic form is, $(D^2 - 2D + 2)\,y = e^x \tan x$

Its A.E. is, $D^2 - 2D + 2 = 0 \Rightarrow D = \frac{2 \pm \sqrt{4-8}}{2}$

$\Rightarrow D = \frac{2 \pm 2i}{2} = 1 \pm i$

$C.F. = e^x\,(c_1 \cos x + c_2 \sin x)$

Let the $P.I. = uy_1 + vy_2$

where $y_1 = e^x \cos x,\ y_2 = e^x \sin x,\ X = e^x \tan x$ ...(1)

Now, $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x \cos x & e^x \sin x \\ e^x(-\sin x + \cos x) & e^x(\sin x + \cos x) \end{vmatrix}$

$= e^{2x}\,[\cos x \sin x + \cos^2 x + \sin^2 x - \sin x \cos x]$

$= e^{2x}$

Now
$$u = -\int \frac{y_2 X}{W}\,dx = -\int \frac{e^x \sin x\, e^x \tan x}{e^{2x}}\,dx$$

$$= -\int \frac{1 - \cos^2 x}{\cos x}\,dx = -\log(\sec x + \tan x) + \sin x$$

and
$$v = \int \frac{y_1 X}{W}\,dx = \int \frac{e^x \cos x\, e^x \tan x}{e^{2x}}\,dx = -\cos x$$

$\therefore$ $P.I. = -e^x \cos x \log(\sec x + \tan x) + e^x \cos x \sin x - e^x \sin x \cos x$

$\therefore$ $y = e^x\,(c_1 \cos x + c_2 \sin x) - e^x \cos x \log(\sec x + \tan x).$

Q 52. Using the method of variation of parameters, solve : $(D^2 + 4)\,y = \sec 2x$. (PTU, May 2004)

Ans. Its A.E. be $D^2 + 4 = 0 \Rightarrow D = \pm 2i$

$\therefore$ $C.F. = C_1 \cos 2x + C_2 \sin 2x$

Let $y_1 = \cos 2x,\ y_2 = \sin 2x,\ X = \sec 2x$

so that its P. I be $uy_1 + vy_2$

$$u = -\int \frac{y_2}{w}\,X\,dx \text{ and } v = \int \frac{y_1}{w}\,X\,dx$$

Now $w = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix} = 2\cos^2 2x + 2\sin^2 2x = 2$

$\therefore$ $u = -\int \frac{\sin 2x \sec 2x}{2}\,dx = \frac{+1}{2} \cdot \frac{1}{2} \log|\cos 2x|$

$v = \int \frac{\cos 2x \sec 2x}{2}\,dx = \frac{1}{2}\int dx = \frac{x}{2}$

$\therefore$ $P.I. = \frac{1}{4} \log|\cos 2x| \cos 2x + \frac{x}{2} \sin 2x$

$\therefore$ $y = C.F. + P.I.$

Q 53. Prove that $\frac{1}{f(D)}\,e^{ax} = \frac{1}{f(a)}\,e^{ax}$; $f(a) \neq 0$. (PTU, Dec. 2004)

Ans. We know that $D\,(e^{ax}) = a\,e^{ax}$

LORDS MODEL TEST PAPER – 2

SECTION – A

Q 1. (a) Verify Euler's theorem for $f(x, y, z) = 3x^2yz + 5xy^2z + 4z^4$.
(b) Define a homogeneous function with the help of one example.
(c) Find the maxima and minima of $f(x, y) = x^3 y^2 (1-x-y)$.
(d) Find the General Solution of the differential equation $(2xy + x^2) y' = 3y^2 + 2xy$
(e) Obtain the general and as well as singular solution of the non-linear equation $y = xy' + (y')^2$
(f) Define order and degree of an ordinary differential equation.
(g) State ratio test for convergence of series.
(h) What do you understand by complementary function? Explain.
(i) Find complementary solution of $9y''' + 3y'' - 5y' + y = 0$.
(j) What do you understand by the uniform convergence of a series? Explain with the help of one example.

SECTION – B

Q 2. Transform $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$ to polar co-ordinates.

Q 3. Discuss the convergence of the series $x + \frac{2^2x^2}{2!} + \frac{3^3x^3}{3!} + \frac{4^4x^4}{4!} + \ldots\ldots$

Q 4. Find the general solution of the equation $y'' + 16y = 32 \sec 2x$, using method of variation of parameters.

Q 5. For what value of 'k' the differential equation

$$\left(1 + e^{kx/y}\right) dx + e^{x/y}\left(1 - \frac{x}{y}\right) dy = 0 \text{ is exact.}$$

Q 6. (a) Use Lagrange's method to find the minimum value of $x^2 + y^2 + z^2$ subject to the conditions $x + y + z = 1$ and $xyz + 1 = 0$.
(b) Prove that the rectangular solid of maximum volume that can be inscribed in a sphere is a cube.

SECTION – C

Q 7. (a) Solve $x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = \log x \sin(\log x)$

(b) Find the volume of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, using integration.

Q 8. Discuss the convergence of the series :

$$1 + \frac{2^2}{3^2} + \frac{2^2.4^2}{3^2.5^2} + \frac{2^2.4^2.6^2}{3^2.5^2.7^2} + \ldots\ldots\infty.$$

Q 9. Solve any two of the following differential equations :

(a) $\frac{dy}{dx} + 2xy = 2e^{-x^2}$

(b) $\frac{dy}{dx} = y \tan x - y^2 \sec x$

(c) $(2x \log x - xy)\, dy + 2y\, dx = 0$.

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FOR NOTES

LORDS MODEL TEST PAPERS

(Unsolved)

LORDS MODEL TEST PAPER – 1

SECTION – A

Q 1. (a) Explain the Lagrange's method of multipliers for maxima and minima.
(b) Define Leibnitz's linear and Bernoulli's equations.
(c) Find solution of the differential equation $y' + y = y^2$.
(d) Explain the convergence and divergence of a series.
(e) Examine the convergence of $\sum\left(\sqrt[3]{n^3+1}-n\right)$.
(f) Write the most general Cauchy's homogeneous linear differential equation.
(g) Solve $x^2 y'' + 4xy' + 2y = 0$
(h) If $z = \sqrt{x^2+y^2}$ and $x^3 + y^3 + 3axy = 5a^2$, find the value of $\frac{dz}{dx}$, when $x = y = a$.
(i) If $u(x, y) = x^y$, find $\frac{\partial^2 u}{\partial x \partial y}$ at (1, 2).
(j) Find the equation of the tangent plane to the surface $xyz = a^3$ at (x_1, y_1, z_1).

SECTION – B

Q 2. Solve the problem $\left(xy^2 - e^{x^{\frac{1}{3}}}\right)dx - x^2y\,dy = 0$

Q 3. Test for convergence of the series $\sum \frac{n^2+1}{n^3+1}$

Q 4. State the integral test for convergence of series and hence discuss convergence of p-series.

Q 5. Solve $x^2\left(\frac{dy}{dx}\right)^2 + 3xy\frac{dy}{dx} + 2y^2 = 0$

Q 6. Prove that $\frac{1}{f(D)}e^{ax} = \frac{1}{f(a)}e^{ax}$; $f(a) \neq 0$.

SECTION – C

Q 7. If two roots of the auxiliary equation $\lambda^2 + a_1\lambda + a_2 = 0$ are real and equal, then prove that $y = (c_1 + c_2 x)e^{\lambda x}$ is a solution of the equation $y'' + a_1y' + a_2y = 0$.

Q 8. State cauchy root test and use it test the convergence of the series : $\sum\left(\frac{n}{n+1}\right)^{n^2}$

Q 9. Test for convergence $\sum \frac{4.7 \ldots\ldots (3n+1)}{1.2 \ldots\ldots n} \cdot x^n$

Thus complete solution be given by

$$y = e^{-x/2}\left[C_1 \cos\left(\frac{\sqrt{3}}{2}x\right) + C_2 \sin\left(\frac{\sqrt{3}}{2}\right)x\right]$$

Q 61. Find Particular integral for $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = e^{-x}$. **(PTU, Dec. 2020)**

Ans. The given diff. eqn in symbolic form be given by

$$(D^2 - 2D + 1)y = e^{-x};\ D = \frac{d}{dx}$$

$$P.I = \frac{1}{D^2 - 2D + 1}e^{-x}$$

$$= \frac{1}{(-1)^2 - 2(-1) + 1}e^{-x} \qquad \left[\because \frac{1}{f(D)}e^{\alpha x} = \frac{e^{\alpha x}}{f(\alpha)};\ \text{where } f(\alpha) \neq 0\right]$$

$$= \frac{1}{4}e^{-x}$$

Q 62. Solve the differential equation $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = xe^{3x} + \sin 2x$. (PTU, Dec. 2020)

Ans. Given differential equation in symbolic form be

$$(D^2 - 3D + 2)\,y = xe^{3x} + \sin 2x;\ D = \frac{d}{dx}$$

Its Anxiliary eqn be given by $m^2 - 3m + 2 = 0$

$\Rightarrow$ $m = 1, 2$

$C.I = C_1e^{x} + C_2e^{2x}$

& $$P.I = \frac{1}{D^2 - 3D + 2}(xe^{3x} + \sin 2x)$$

$$= \frac{1}{D^2 - 3D + 2}xe^{3x} + \frac{1}{D^2 - 3D + 2}\sin 2x$$

$$= e^{3x}\frac{1}{(D+3)^2 - 3(D+3) + 2}x + \frac{1}{D^2 - 3D + 2}\sin 2x \qquad \left(\because \frac{1}{f(D)}e^{ax}V = e^{ax}\frac{1}{f(D+a)}V\right)$$

$$= e^{3x}\frac{1}{D^2 + 3D + 2}x + \frac{1}{-2^2 - 3D + 2}(\sin 2x) \qquad \left(\because \frac{1}{f(D^2)}\sin ax = \frac{1}{f(-a^2)}\sin ax\right)$$

$$= e^{3x}\frac{1}{2\left[1 + \frac{D^2 + 3D}{2}\right]}x + \frac{1}{-3D - 2}\sin 2x$$

$$= \frac{e^{3x}}{2}\left[1 + \frac{D^2 + 3D}{2}\right]^{-1}(x) - \frac{3D - 2}{9D^2 - 4}(\sin 2x)$$

$$= \frac{e^{3x}}{2}\left[1 - \frac{D^2 + 3D}{2}\right](x) - \frac{(3D - 2)}{9(-2)^2 - 4}(\sin 2x)$$

$$= \frac{e^{3x}}{2}\left[x - \frac{1}{2}(0 + 3)\right] + \frac{1}{40}(6\cos 2x - 2\sin 2x)$$

$$= \frac{e^{3x}}{2}\left(x - \frac{3}{2}\right) + \frac{1}{40}(6\cos 2x - 2\sin 2x)$$

Thus $C.S. = C.F + P.I$

$$= C_1e^{x} + C_2e^{2x} + \frac{e^{3x}}{2}\left(x - \frac{3}{2}\right) + \frac{1}{20}(3\cos 2x - \sin 2x)$$

Q 63. Solve $(1+x)^2\frac{d^2y}{dx^2} + (1+x)\frac{dy}{dx} + y = \cos\ln(1+x)$. **(PTU, Dec. 2020)**

Ans. Given differential eqn in symbolic form can be written as ;

$[(1+x)^2D^2 + (1+x)D + 1]\,y = \cos\ln(1+x)$ ...(1)

where $D = \frac{d}{dx}$

eqn (1) is of legendre's form

put $(1+x) = e^z \Rightarrow z = \ln(1+x)$

$\therefore$ $(1+x)D = \theta;\ (1+x)^2D^2 = \theta(\theta - 1);\ \theta = \frac{d}{dz}$

Then eqn (1) becomes;

$[\theta(\theta - 1) + \theta + 1]\,y = \cos z$

$\Rightarrow$ $[\theta^2 + 1]\,y = \cos z$

Thus, A.E. be given by $\theta^2 + 1 = 0 \Rightarrow \theta = \pm i$

$C.F. = C_1\cos z + C_2\sin z$

& $$P.I = \frac{1}{\theta^2 + 1}\cos z = z\frac{1}{2\theta}\cos z$$

$$= \frac{z}{2}\sin z$$

$$\left[\because \frac{1}{f(D^2)}\cos ax = \frac{1}{f(-a^2)}\cos ax,\ f(-a^2) \neq 0\right]$$

If $f(-a^2) = 0$ then $\frac{1}{f(D)}\cos ax = x.\frac{1}{f'(D)}\cos ax$

$$C.S = y = C.F + P.I$$

$$= C_1\cos z + C_2\sin z + \frac{z}{2}\cos z$$

$$= C_1\cos\ln(1+x) + C_2\sin\ln(1+x) + \frac{\ln(1+x)}{2}\cos\ln(1+x)$$

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